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FOREIGN LOANS AND ECONOMIC DEVELOPMENT

PART II

THE FIXED ANNUITY PRINCIPLE

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CONTENTS

I. Introduction:

In Part I of this memo.[1] we dealt with the problem of assessing the advantages of foreign loans, using a model suggested by Qayum.[2] Three main problems have been considered:

1. Discussion of the advantage criteria, with special emphasis on long-term development.
2. Correction of Qayum's formula.
3. Introduction of the replacement period as a parameter in evaluating foreign loans.

The main outcome of that analysis was that the decisive factor in determining whether a loan has a negative or a positive effect on the debtor country's capital, and hence income, at the end of the repayment period, is the difference between the ratio of the marginal savings rate to the capital coefficient, and the rate of interest on the loan. Whenever that ratio exceeds the interest rate, a positive effect can be realized, and vice versa. On the other hand, the elongation of the repayment period was not always a blessing. Its role is to magnify the effect of the loan, whether positive or negative. Grace periods of an equal length were found to be more beneficial in introducing extra gains which might, in some cases, outweigh any initial losses.

Taking account of the orders of magnitude of the parameters[3] σ and γ, normally prevailing in developing countries, one should expect relatively low rates of interest if resort to foreign loans would contribute at all to the process of long-term development. Moreover, if the interest rate is very close to the ratio σ/γ, the repayment period should be extended considerably over time in order to recover the whole benefit of the loan at some given date. In some cases, the repayment period has to be several times as long as the period required for replacing the assets financed through the loan.

We have concluded our previous paper by asserting the need for:

1. Reconsidering the behaviour of the parameters, and
2. Studying the possibility of adopting the fixed instalment principle.

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(1) Memo. 779, Part I. I.N.P., June 1967
(2) Memo. 570, I.N.P., May, 1965
(3) For the system of notation, see Memo. 779, Part I, pp. 4-5

We shall start by considering this latter problem in order to see how far our previous conclusions would be affected by a change of the repayment conditions. Later on we shall consider the policy implications of alternative advantage criteria, and their effects on the values of the parameters

II. The Fixed Annuity Formula:

In Part I, we have given the formula for a fixed annuity based on compound interest. Thus, if the Annuity is A, its ratio to the loan is b:

$$b = A/L, \qquad \text{or,} \qquad A = bL \tag{1}$$

The value of b under the compound interest rule, b_c is [4]

$$b_t = b_c = \frac{r.(1+r)^e}{(1+r)^e - 1} \qquad (t = 1.....e) \tag{2}$$

where r is the interest rate and e is the repayment period. A more plausible assumption is to assume that repayment follows the simple interest rule:

Suppose that a loan L received in year 0 is to be repaid over e years at a simple rate of interest r. Repayment is made by means of a fixed annuity A, paid at the end of each year. To determine A, we start by considering all flows envolved at a given point of time, e.g., at the end of the repayment period. Thus the inflow will be:

$$L\,(1 + er)$$

The repayment made in the t-th year will reach the value

$$A\left[1 + r\,(e - t)\right]$$

at the same point of time. Hence the total outflow evaluated at the end of year e will be

$$\sum_{t=1}^{e} A\left[1 + r\,(e - t)\right] = A\left[e + r\sum_{t=1}^{e}(e - t)\right]$$
$$= eA + rA\sum_{j=0}^{e-1} j = eA\left[1 + \frac{r\,(e-1)}{2}\right]$$

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(4) There is a printing error in this formula as given in memo. 779 Part I, p. 7 The factor r is missing from the numerator.

Equating the two flows, and substituting (1), we can easily find that:

$$b = \frac{2\,(1 + er)}{e\left[2 + r\,(e - 1)\right]} \qquad (3)$$

It has been shown that if the principal is to be repaid at equal instalments while interest is charged on the remainder, the annuity will be decreasing. Its value was found to be:(5)

$$b_t = A_t/L = \frac{1}{e}\left[1 + (e - t + 1)\,r\right] \qquad (I/3)$$

For year t = 1, this value is greater than b since:

$$b_1/b = \frac{1}{e}\,(1 + er)\;\frac{e\left[2 + r(e - 1)\right]}{2\,(1 + er)} = 1 + r\left(\frac{e - 1}{2}\right) \geqslant 1$$

The equality sign holds only when e = 1, while the inequality holds for larger values of e, so long as r is positive. On the other hand, the annuity b_t will be decreasing at a rate equal to r/e. This means that its ratio to a growing income will be falling more rapidly. In other words, the fixed annuity principle is less unfavourable than the decreasing annuity approach, to the debtor country.

Besides, the total repayments made on the basis of the fixed annuity principle is smaller than that following the decreasing annuity rule:

$$\sum_{t=1}^{e} b_t = \sum_{t=1}^{e} \frac{1}{e}\left[1 + (e + 1)\,r - tr\right] = 1 + \frac{r\,(e + 1)}{2}$$

$$\sum_{t=1}^{e} b = e \cdot \frac{2\,(1 + er)}{e\left[2 + r(e - 1)\right]} = 1 + \frac{r\,(e + 1)}{2 + r\,(e - 1)}$$

Thus:

$$\sum b_t > eb, \qquad \text{for any } e > 1$$

which means that the outflow of capital will be larger for the variable annuity approach.

It would be also expected that the capitalized value of the variable annuities is larger than that of the fixed annuity, which was found to be (1 + er). Now,

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(5) Equations quoted from 779, Part I, will be given their numbers preceded with I.

$$\sum_{t=1}^{e} b_t \left[1 + (e - t)\, r\right] = \frac{1}{e} \sum_{j=0}^{e-1} (1 + r + jr)(1 + jr)$$

$$= (1 + r) + \frac{r\,(2 + r)(e - 1)}{2} + \frac{r^2\,(e - 1)(2e - 1)}{6}$$

$$= 1 + er + \frac{(e - 1)(e + 1)}{3}\, r^2 > (1 + er)$$

Thus the fixed annuity approach seems to be more favourable to the debtor country, on either of the three aspects considered:

1. The value of the annuity is smaller at the beginning, hence its burden on initial lower incomes is smaller.
2. It leads to a smaller capital outflow.
3. The capitalized value of the outflow is also smaller.

It might be argued, therefore, that the stringent conditions obtained before could be somewhat relaxed if we accept the fixed annuity approach. To the answer of this question, we shall address our-selves in the following sections.

III. The Impact of a Single Loan:

Let us start by assuming the set of assumptions (1) - (5), section V, memo 779, Part I (pp. 14-15). Assumption (6) is to be replaced by the fixed annuity assumption. The base year values are given:

$$Y_0, \qquad S_0 = sY_0, \qquad I_0 = (s + \lambda)\, Y_0$$

For year 1, the values of income and savings are the same as before:

$$Y_1 = \left(1 + \frac{s + \lambda}{\gamma}\right) Y_0 \tag{I/10}$$

$$S_1 = \left(d + \lambda \frac{\sigma}{\gamma}\right) Y_0 = c\, Y_0 \tag{I/12}$$

where

$$d = s\left(1 + \frac{\sigma}{\gamma}\right) \tag{I/32}$$

But instead of (I/13), we have:

$$I_1 = S_1 - A = \left(1 + \frac{\sigma}{\gamma}\right) S_0 + \left(\frac{\sigma}{\gamma} - b\right) L$$

$$\text{or,} \quad I_1 = \left[d + \lambda \left(\frac{\sigma}{\gamma} - b\right)\right] Y_0 \tag{4}$$

For the following years, t = 2, ...,e, we have the following system of equations:

$$Y_t = Y_{t-1} + \frac{1}{\gamma}\, I_{t-1} \tag{5}$$

$$S_t = S_1 + \sigma\,(Y_t - Y_1) \tag{6}$$

$$I_t = S_t - A = \sigma(Y_t - Y_1) + I_1 \tag{7}$$

Equations (5) and (6) are the same as (I/15) and (I/16), while (7) differs from (I/17) in so far as the value of annuity is concerned. Substituting (7) in (5) we obtain:

$$Y_t = (1 + \frac{\sigma}{\gamma})\, Y_{t-1} + \frac{1}{\gamma}\,(I_1 - \sigma Y_1) \qquad (8)$$

where I_1 and Y_1 are given by (4) and (I/10). The solution of this first-order difference equation is:

$$Y_t = (1 + \frac{\sigma}{\gamma})^{t-1} \cdot (\frac{1}{\sigma} I_1) + (Y_1 - \frac{1}{\sigma} I_t)$$

or,

$$Y_t \; (1 + \frac{\sigma}{\gamma})^{t-1}\Big[\frac{s}{\sigma}\,(1 + \frac{\sigma}{\gamma}) + \frac{\lambda}{\sigma}\,(\frac{\sigma}{\gamma} - b)\Big]\, Y_o + \frac{1}{\sigma}\;(\sigma - s + \lambda b) Y_o \qquad (9)$$

If no loan was taken, the GDP. can be calculated from this last formula by putting $\lambda = 0$:

$$Y'_t = (1 + \frac{\sigma}{\gamma})^t\, \frac{s}{\sigma}\; Y_o - (1 - \frac{s}{\sigma})\; Y_o \qquad (10)$$

Thus for any year $t = 2, \ldots, e, e + 1$, we have:

$$Y_t - Y'_t = (1 + \frac{\sigma}{\gamma})^{t-1}\; \frac{\lambda}{\sigma}\; (\frac{\sigma}{\gamma} - b) Y_o + \frac{\lambda}{\sigma}\, b\, Y_o \qquad (11)$$

In particular, $V = Y_{e+1} - Y'_{e+1}$ defined by (I/29) is equal to:

$$V = \Big[E\,(\frac{\sigma}{\gamma} - b) + b\Big]\frac{\lambda}{\sigma}\; Y_o \qquad (12)$$

where E is defined according to (I/32)

$$E = (1 + \frac{\sigma}{\gamma})^e$$

The criteria ε and δ defined by (I/28) are:

$$\varepsilon = \frac{\gamma}{\lambda}\;\frac{V}{Y_o} = \Big[E\,(\frac{\sigma}{\gamma} - b) + b\Big]\frac{\gamma}{\sigma}$$

Hence:

$$\varepsilon = \frac{\gamma}{\sigma}\,(E - 1)\,(\frac{\sigma}{\gamma} - b) + 1 \qquad (13)$$

and, $$\delta = \varepsilon - 1 = \frac{\gamma}{\sigma}\,(E - 1)\,(\frac{\sigma}{\gamma} - b) \qquad (14)$$

For given values of $\frac{\sigma}{\gamma}$ and r, these expressions are increasing functions of e. As e increases the value of b diminishes. It might be possible that for small values of e the factor $(\frac{\sigma}{\gamma} - b)$ is negative. As e increases it would change signs, and its value will be increasing. At the same time the value of the positive factor multiplying it, (E - 1) will be increasing. This result is different from our earlier finding, for the case of a decreasing annuity, but it is in line with the findings of Qayum and of I.B.R.D. experts.(6)

Again, since b is an increasing function of r, both ϵ and δ will be decreasing as r increases. On the other hand, we can write:

$$\frac{\gamma}{\sigma}(E - 1) = \sum_{0}^{e-1} (1 + \frac{\sigma}{\gamma})^t$$

which is an increasing function of $\frac{\sigma}{\gamma}$. The same is also true for the factor $(\frac{\sigma}{\gamma} - b)$, which means that both criteria are increasing functions of σ, and decreasing functions of γ. Let us now turn to the other parameters envolved, before discussing the implications of the alternative criteria.

IV. The Formulae with Lags and Replacement:

Suppose that projects financed by the loan have a gestation lag of f years, which means that they start production in year f + 1. Further, a grace period of a length h is accorded. As had been suggested in Part I, we consider that the life time of the projects financed by the loan is k, which might be equal to or different from e. To calculate the values of the alternative criteria, we subdivide investment in any year t into two parts, one due to incomes derived from the loans, j", and the remainder J':

$$I_t = J'_t + J''_t \qquad (15)$$

The part J' changes its composition starting year h + 1. For the first h years we have:

$$J'_t = s\,(1 + \frac{\sigma}{\gamma})^t\ Y_o, \qquad (t = 1,\ldots,h) \qquad (16)$$

In year h + 1 it becomes equal to:

$$J'_{h=1} = \left[s\ (1 + \frac{\sigma}{\gamma})^{h+1} - \lambda b\right] Y_o = (dH - \lambda b)\ Y_o$$

For the repayment period we have:

(6) See, e.g., G.M. Alter:" The Servicing of Foreign Capital Inflow by under developed countries". in, H.S. Ellis & H.C. Wallich (ed): <u>Economic Development for Latin Amercia</u>; Micmillan, N. Y., 1961

$$J'_t = J'_{h+1} \left(1 + \frac{\sigma}{\gamma}\right)^{t-h-1} = (dH - \lambda b)\left(1 + \frac{\sigma}{\gamma}\right)^{t-h-1} Y_o$$

which is true for $t = h + 1, \ldots, \tau \leqslant h + e$. The cumulative investments of type J' can be found by summation:

$$\sum_1^{\tau} J'_t = \sum_1^{h} J'_t + \sum_{h+1}^{\tau} J'_t$$
$$= \frac{\gamma}{\sigma}\left[d\ (H - 1) + (dH - \lambda b)\left(\frac{T}{H} - 1\right)\right] Y_o$$

where H is defined by (I/32), and T is defined in a similar manner:

$$T = \left(1 + \frac{\sigma}{\gamma}\right)^{\tau} \tag{17}$$

In other words:

$$\sum_1^{\tau} J'_t = \frac{\gamma}{\sigma}\left[d\ (T - 1) - \lambda b\ \left(\frac{T}{H} - 1\right)\right] Y_o \tag{18}$$

The investments financed by savings generated from loan incomes start in year f + 1,

$$J''_{f+1} = \lambda \frac{\sigma}{\gamma} Y_o$$

For the following years,

$$J''_t = \lambda \frac{\sigma}{\gamma}\left(1 + \frac{\sigma}{\gamma}\right)^{t-f-1} Y_o \qquad (t \geqslant f+1)$$

Hence,

$$\sum_1^{\tau} J''_t = \sum_{f+1}^{\tau} J''_t = \lambda\ \left(\frac{T}{F} - 1\right) Y_o \tag{19}$$

It follows that the sum-total of investments during the τ years is:

$$\sum_1^{\tau} I_t = \frac{\gamma}{\sigma}\left[d\ (T - 1) + \lambda \frac{\sigma}{\gamma}\left(\frac{T}{F} - 1\right) - \lambda b\ \left(\frac{T}{H} - 1\right)\right] Y_o \tag{20}$$

Starting year h + e + 1, repayments disappear which means that:

$$I_t = I_{h+e+1} \left(1 + \frac{\sigma}{\gamma}\right)^{t-h-e-1}$$

It can be seen that:

$$I_{h+e+1} = I_{h+e} \left(1 + \frac{\sigma}{\gamma}\right) + \lambda b Y_o$$

To calculate I_{h+e}, we calculate its two components:

$$J'_{h+e} = (dH - \lambda b)\ \left(1 + \frac{\sigma}{\gamma}\right)^{e-1} Y_o$$

$$J''_{h+e} = \lambda \frac{\sigma}{\gamma}(1 + \frac{\sigma}{\gamma})^{h+e-f-1} \; Y_o$$

Substituting in the above expressions we obtain:

$$I_{h+e+1} = (dH + \lambda \frac{\sigma}{\gamma}\frac{H}{F} - \lambda b)\;(1 + \frac{\sigma}{\gamma})^e \; Y_o + \lambda b \; Y_o$$

Hence for $t \geqslant h + e + 1$

$$I_t = (dH + \lambda \frac{\sigma}{\gamma}\frac{H}{F} - \lambda b)(1 + \frac{\sigma}{\gamma})^{t-h-1} \; Y_o + \lambda b(1 + \frac{\sigma}{\gamma})^{t-h-e-1} \; Y_o \qquad (21)$$

and,

$$\sum_{e+h+1}^{\tau} I_t = \frac{\gamma}{\sigma}\Big[d(T - HE) + \lambda \frac{\sigma}{\gamma}(\frac{T}{F} - E^*) + \lambda b(\frac{T}{HE} - \frac{T}{H} + E-1)\Big] \; Y_o$$

where, as before, $E^* = HE/F$. In order to obtain the cumulation of investment starting year 1, we have to add to this last sum the cumulation (20) up to $\tau = h+e$. Hence

$$\sum_{1}^{h+e} I_t = \frac{\gamma}{\sigma}\Big[d\;(HE - 1) + \lambda \frac{\sigma}{\gamma}(E^* - 1) - \lambda\; b\; (E - 1)\Big] \; Y_o$$

It follows that, for $\tau > h + e$:

$$\sum_{1}^{\tau} I_t = \frac{\gamma}{\sigma}\Big[\;d\;(T - 1) + \lambda \frac{\sigma}{\gamma}(\frac{T}{F} - 1) - \lambda\; b\; \frac{T}{HE}\;(E - 1)\Big] Y_o \qquad (22)$$

Finally, in the case of no loans we have for any t:

$$I'_t = d\;(1 + \frac{\sigma}{\gamma})^{t-1} \qquad (23)$$

Hence, for any τ,

$$\sum_{1}^{\tau} I'_t = d\;\frac{\gamma}{\sigma}\;(T - 1) \qquad (24)$$

It follows that:

$$\Delta_\tau = \sum_{1}^{\tau}\;(I_t - I'_t) \qquad (25)$$

has the following values for the different values of τ:

(1) <u>$\tau = f + k < h + e$:</u> Using (20) and (24),

$$\Delta_{f+k} = \lambda \frac{\gamma}{\sigma}\Big[\frac{\sigma}{\gamma}(K - 1) - b\;(K^* - 1\;)\Big] \; Y_o \qquad (26)$$

where, as in (I/61), $K^* = FK/H$,

(2) $\tau = f + k = h + e$: Using (20) and (24):

$$\Delta_{h+e} = \lambda \frac{\gamma}{\sigma}\left[\frac{\sigma}{\gamma}(E^{*}-1) - b\ (E - 1)\right] Y_o \qquad (27)$$

(3) $\tau = f + k > h + e$: From (22) and (24):

$$\Delta_{f+k} = \lambda \frac{\gamma}{\sigma}\left[\frac{\sigma}{\gamma}\ (K - 1) - b\,\frac{K^{*}}{E}\ (E - 1)\right] Y_o \qquad (28)$$

It is easy to derive the corresponding formulae for the cases of no lags. In such cases, both f and h are zero, hence $H = F = 1$, and $E^{*} = E$, while $K^{*} = K$.

The criteria by which we judge the adventage of the loan are based on assigning some minimum value (e.g., 0 or 1) to ($\Delta / \lambda \frac{\sigma}{\gamma} Y_o$). In other words, the expression between brackets in (26) - (28) should be not less than some given constant. Solving this condition for the amnuity b, and noticing that it should be non-negative, we obtain:

$$0 \leqslant b \leqslant \varphi \qquad (29)$$

where φ is some given function of some or all parameters except r.

According to (3), the annuity b is a decreasing function of e and an increasing function of r.

$$\frac{\partial b}{\partial e} = \frac{-\left[4\ (e^2\ r^2 + 1) + 2r\ (2e - 1)\right]}{e^2\left[2 + (e - 1)\ r\right]^2}$$

$$\frac{\partial b}{\partial r} = \frac{2\ (e + 1)}{e\ \left[2 + (e - 1)\ r\right]^2}$$

For all non-negative values of r and e, the first expression is negative while the second is positive. It follows that:

$$\frac{de}{dr} = -\frac{\partial e}{\partial b} \times \frac{\partial b}{\partial r}$$

is always positive.

Thus for a given e the minimum value of b occurs at $r = 0$, in which case $b = \frac{1}{e}$. This value decreases as e increases. From (29), it follows that e should fall within the range defined by

$$0 \leqslant \frac{1}{e} \leqslant \varphi$$

In other words

$$\frac{1}{\varphi} \leqslant e \leqslant \infty \qquad (30)$$

so long as $r = o$. On the other hand for an infinite value of r, equation (3) shows that the annuity will be $b \doteq 2/(e-1)$. Substituting in (29), then,

$$0 \leqslant \frac{2}{e-1} \leqslant \varphi$$

or,

$$\frac{2}{\varphi} + 1 \leqslant e \leqslant \infty$$

It follows that the range of variation of e such that(29) is satisfied would be subdivided into two parts:

(1) The range defined by the last expression for any value of r, namely:

$$\frac{2}{\varphi} + 1 \leqslant e \leqslant \infty, \qquad \text{for} \qquad 0 \leqslant r \leqslant \infty \qquad (31)$$

(2) The range lying between (30) and (31), in which case **r** should not exceed an upper limit after which (29) would be violated. To find this limit we solve (3) for r;

$$r = \frac{2\ (be - 1)}{e\left[\,2 - (e - 1)\,b\,\right]} \qquad (32)$$

Since, as shown above, b is an increasing function of r, the upper bound on b defines the upper bound on r. In other words, we can substitute for b in (32) to define the upper limit $\bar{r}$, given e. This means that:

$$\frac{1}{\varphi} \leqslant e \leqslant \frac{2}{\varphi} + 1, \qquad \text{for,} \quad 0 \leqslant \bar{r} \leqslant \frac{2\,(e\varphi - 1)}{e\left[\,2 - (e - 1)\,\varphi\right]} \qquad (33)$$

The absolute minimum is at $r = 0$ and $e = 1/\varphi$. When e reaches its upper bound, the denominator of the upper bound of r is zero. Hence $\bar{r}$ becomes infinite and we move into the region defined by (31), which admits any non-negative value for r. We have only, therefore, to calculate the maximum r at each value of e defined by (33). Alternatively we solve the quadratic obtained from the upper bound of r to obtain the corresponding minimum of e:

$$r\varphi e^2 - (2r + \varphi r - 2\varphi) - 2 \geqslant 0 \qquad (34)$$

For a given rate of interest $r = r^*$, we calculate the positive root of this quadratic, $e^* = e\ (r^*)$. Hence, for any value r^* of r,

$$e^* \leqslant e \leqslant \infty \qquad (35)$$

V. Advantage Criteria in the Absence of Lags:

We have already obtained expressions for the two criteria:

$$\delta = \Delta_e / \lambda Y_o \qquad \epsilon = \delta + 1$$

They are measured at the end of the repayment period e, assuming that $f = h = c = o$. Let us consider them in turn in the light of the remarks of the last section.

(1) $\delta \geqslant 0$, or $\epsilon \geqslant 1$:

This condition requires that by the end of the repayment period, the capital stock should exceed what could have been obtained in the absence of loans by an amount at least equal to the loan itself. Using (14), we have:

$$b \leqslant \frac{\sigma}{\gamma} \qquad \text{or,} \qquad \sigma \geqslant b\gamma \qquad (36)$$

In other words, given the output $N = L/\gamma$ due to the loan, the fixed annuity should satisfy the condition.

$$A \leqslant \sigma N \qquad \text{or,} \qquad \sigma \geqslant A/N$$

The meaning of this condition is evident. If the repayment process is not to encroach upon the country's resources, it has to be provided for, out of the extra incomes due to the loan itself. Thus if $\gamma = 3$, an annuity equal to 12% of the value of the loan would require a m.p.s. of at least 36%.

To study the implications of this condition for the terms of the loan, we notice that $\varphi = \frac{\sigma}{\gamma}$. Hence r can take any positive value for any e satisfying (31), namely:

$$e \geqslant 2\frac{\gamma}{\sigma} + 1 \qquad \text{for any } r \geqslant 0 \qquad (37)$$

For the range defined by (33), the rate of interest is restricted:

$$\frac{\gamma}{\sigma} \leqslant e \leqslant 2\frac{\gamma}{\sigma} + 1, \qquad \text{for} \quad 0 \leqslant r \leqslant \frac{2\left(e\frac{\gamma}{\sigma} - 1\right)}{e\left[2 - (e-1)\frac{\gamma}{\sigma}\right]} \qquad (38)$$

Comparing (37) and (38), with (I/49):

$$\frac{\gamma}{\sigma} \leqslant e \leqslant \infty \qquad \text{for which} \qquad 0 \leqslant r \leqslant \frac{\sigma}{\gamma}$$

we find that the two principles concide in the lower limits: $e = \frac{\gamma}{\sigma}$, with r = o. As e increases, the upper limit of r rises quicker in the fixed annuity case. To find the value of e for which $r \leqslant \sigma/\gamma$, the alsolute upper bound in the decreasing annuity case, we substitute in (34):

$$\left(\frac{\sigma}{\gamma}\right)^2 e^2 - \left(\frac{\sigma}{\gamma}\right)^2 e - 2 \geqslant 0$$

This gives:

$$e^* = \frac{\gamma}{\sigma}\sqrt{2}\left[1 + \frac{\sqrt{2}}{4}\frac{\sigma}{\gamma} + \frac{1}{8}\left(\frac{\sigma}{\gamma}\right)^2\right] \leqslant e \leqslant \infty \qquad (39)$$

as the range of variation of e.

Example: Let $\frac{\sigma}{\gamma} = 0.05$ The alsolute minimum of e is 20, at which r = o If r = 5% ,the minimum e is, by (39), equal to 33.9. This could not be accepted in the decreasing annuity case unless e were infinite.

(2) $\varepsilon \geqslant 0$, or $\delta \geqslant -1$:

This is a milder condition,ensuring that the country would not lose any of its self-generated capital, though not necessarily retaining any net additions as a result of the loan. Solving this inequality, using (13) we obtain the condition.

$$\varphi = \frac{\sigma}{\gamma}\left(\frac{E}{E-1}\right), \qquad \text{or,} \qquad b \leqslant \frac{\sigma}{\gamma}\left(\frac{E}{E-1}\right) \qquad (40)$$

This raises the upper limit of b above the level determined by (36). However, (40) is quite an envolved expression in both e and σ/γ. Let us therefore introduce the approximation:

$$E \geqslant 1 + e\frac{\sigma}{\gamma} \qquad (41)$$

which means that

$$\varphi \leqslant \frac{1}{e} + \frac{\sigma}{\gamma}$$

Substituting in (40), it follows that:

$$\sigma \geqslant (b - \frac{1}{e})\gamma \qquad \text{(app)} \qquad (42)$$

which reduces the lower limit of the m.p.s. below the level given by (36).

Substituting in (33), and rearranging terms we can easily find that, $1 \leqslant e\varphi \leqslant 2 + \varphi$, which means that:

$$0 \leqslant e \leqslant \frac{\gamma}{\sigma} + 2 - 2\frac{\sigma}{\gamma} \qquad \text{(app.)} \qquad (43)$$

Compared with (38), both bounds defined by (43) are smaller as a result of the relaxation of the condition on ε. Given a value of e within the above range, we can calculate the exact value of φ from (40), then substitute in (33) to determine the upper limit of r. However the expression for the limit (42) of σ, has to be approximative. It is exact only in the case e = 1.

In the present case, the same lower bound on e is defined as in the decreasing annuity case, equation (I/47). But the upper limit of r rises very rapidly to infinty. Again the flexibility of the fixed annuity principle is evident.

VI. Advantage Criteria in The Case of Lags:

In this case, the criteria take the more general form:

$$\eta = \Delta_{h+e} / \lambda Y_o \qquad \mu = \eta + 1$$

Using (27), we find that:

$$\eta = (E^* - E) + \frac{\gamma}{\sigma}(E - 1)\left(\frac{\sigma}{\gamma} - b\right)$$

In other words:

$$\eta = \delta + m \qquad \mu = \varepsilon + m \qquad (44)$$

where, as before,

$$m = (E^* - E) = \left(\frac{H}{F} - 1\right)E = (C - 1)E \qquad (45)$$

and

$$C = \left(1 + \frac{\sigma}{\gamma}\right)^c, \qquad c = h - f \qquad (46)$$

Equation (44) is the same as (I/41) for the decreasing annuity case. It follows that:

$$\mu = CE - \frac{\gamma}{\sigma} b(E - 1) \qquad (47)$$

Given the values of e, r and σ / γ, we can obtain the value of C for any given μ; e.g., 0 or -1. Solving for b, we obtain:

$$b = \frac{\sigma}{\gamma} \left(\frac{CE - \mu}{E - 1} \right) \qquad (48)$$

If h = f, then C = 1, and $\mu = \varepsilon$, which means that:

$$b = \frac{\sigma}{\gamma} \left(\frac{E - \varepsilon}{E - 1} \right) \qquad (49)$$

As a special case, we have h = f = o in the absence of lags, and (49) gives (36) and (40) for ε = 1 or 0.

Denoting (48) by φ, we can substitute in (33). Solving for μ we find that:

$$CE - 1 \geqslant \mu \geqslant CE - 2 - \frac{2}{2 - 1} \qquad (50)$$

In other words the criterion μ changes in a range of approximately one unity as r changes from zero to infinity, for a given e. An approximate solution for e can be obtained as follows:

$$\left(\frac{1 + \mu}{C} - 1 \right) \frac{\gamma}{\sigma} \leqslant e \leqslant \left(\frac{2 + \mu}{C} - 1 \right) \frac{\gamma}{\sigma} + (2 - \mu) \qquad (51)$$

If we put C = 1, and $\mu = \varepsilon = 1$, we obtain (38).

If $\mu = \varepsilon = 0$, we obtain (43)

To achieve a certain value of μ, we choose a value for e within the range (51). Then the corresponding rate of interest can be calculated by substituting (48) in (32).

$$r = \frac{2e\,(\sigma/\gamma)\,(CE - \mu) - 2\,(E - 1)}{2e\,(E - 1) - e\,(e - 1)\,\frac{\sigma}{\gamma}\,(EC - \mu)} \qquad (52)$$

If the resulting value of r is inacceptable, we have to change the value of e in the necessary direction. Let us turn now to specific values of the criterion:

(1) $\eta \geqslant 0$; or $\mu \geqslant 1$:

We have by (48):

$$b \leqslant \frac{\sigma}{\gamma} \left(\frac{CE - 1}{E - 1}\right) \qquad (53)$$

The range of minimum values of e for r changing from 0 to infinity is defined by (51):

$$\left(\frac{2}{C} - 1\right)\frac{\gamma}{\sigma} \leqslant e \leqslant \left(\frac{3}{C} - 1\right)\frac{\gamma}{\sigma} + 1 \qquad (54)$$

Given the loan conditions, we can solve (53), for the lower limit of σ:

$$\sigma \geqslant b \left(1 - \frac{c}{c + e}\right)\gamma \qquad \text{(app.)} \qquad (55)$$

For c = o, we obtain the same formulae derived for $\mathcal{E} = 1$ in the previous section.

(2) $\mu \geqslant 0$, or, $\eta \geqslant -1$:

We have by (48)

$$b \leqslant \frac{\sigma}{\gamma} \left(\frac{CE}{E-1}\right) \qquad (56)$$

This is equal to C times the limit defined by (40) in the no-lags case. The approximate solution for σ is:

$$\sigma \geqslant \left(\frac{b}{C} - \frac{1}{e}\right)\gamma \qquad \text{(app.)} \qquad (57)$$

The range of variation (51) of e is:

$$\left(\frac{1}{C} - 1\right)\frac{\gamma}{\sigma} \leqslant e \leqslant \left(\frac{2}{C} - 1\right)\frac{\gamma}{\sigma} + 2 \qquad (58)$$

For relatively large values of C, (58) will become meaningless. If C is greater than 1 but smaller than 2, the lower bound will be zero. For values of C greater than 2, the peresent case will give permissible values of r and e equal to or greater than zero which are consistent with the condition that μ should not fall below zero.

VII. The Replacement Rule:

According to this rule, it is required that the extra capital generated over and obove the loan, would be equal to zero by the end of the life of the projects financed by the loan. Measured from the base year, this life ends in year f + k, where f is the gestation period, and k the replacement period. This means that:

$$\Delta_{f + k} = 0$$

where Δ is given by (26) - (28). We have already investigated the case : $f + k = h + e$, or $k = e + c$, for which:

$$K = EC, \qquad K^{*} = \frac{K}{C} = E \tag{59}$$

Let us consider the other two cases:

(1) <u>$k < e + c$:</u>

In this case, the end of the life of the projects occurs before the retirement of the loan. This is a situation which might occur with respect to short-term types of finance, rather than long-term needs. The zero solution of (26) gives:

$$b = \frac{\sigma}{\gamma}\left(\frac{K - 1}{K^{*} - 1}\right) = \frac{\sigma}{\gamma}\left(\frac{CK^{*} - 1}{K^{*} - 1}\right) \tag{60}$$

Given $\frac{\sigma}{\gamma}$, k and c, we can calculate:

$$\Theta = \frac{Ck^{*} - 1}{K^{*} - 1} = C + \left(\frac{C - 1}{K^{*} - 1}\right) \tag{61}$$

and hence b. Further the advantage criterion is:

$$\mu = \theta + (C - \theta) E = C + (1 - C) \left(\frac{E - 1}{K^{*} - 1}\right) \tag{62}$$

In the present case:

$$K^{*} = \frac{K}{C} < E \qquad \therefore \quad \frac{E - 1}{K^{*} - 1} > 1$$

It follows that,

$$\left.\begin{array}{lll} \text{if } C < 1; & \theta < C < 1, & \mu > 1 \\ \text{if } C = 1; & \theta = C = 1, & \mu = 1 \\ \text{if } C > 1; & \theta > C > 1, & \mu < 1 \end{array}\right\} \tag{63}$$

For example, when $C < 1$, we have $m = (C - 1) E < 0$. At the same time, $\mu > 1$, which means that $\varepsilon > (1-m) > 1$. In other words, the loss due to the small value of C has to be compensated by choosing e large enough. The high value of the criterion ensures regaining the value of the loan before the end of the repayment period. Similar remarks can be made for the other two cases.

To obtain the critical value of σ we have to solve (60), using the familiar approximations:

$$b = \frac{\sigma}{\gamma} \cdot \left(\frac{k\,\sigma/\gamma}{k^{*}\sigma/\gamma}\right)$$

Hence,

$$\sigma = b\left(1 - \frac{c}{k}\right)\gamma \qquad \text{(app.)} \qquad (64)$$

The factor φ introduced in (29) is:

$$\varphi = \frac{\sigma}{\gamma}\,\theta \qquad (65)$$

Substituting in (33), we obtain the boundaries:

$$\frac{1}{\theta} \cdot \frac{\gamma}{\sigma} \leqslant e \leqslant \frac{2}{\theta} \cdot \frac{\gamma}{\sigma} + 1 \qquad (66)$$

corresponding to:

$$0 \leqslant r \leqslant \frac{2\left(e\frac{\sigma}{\gamma}\theta + 1\right)}{e\left[2 - (e-1)\frac{\sigma}{\gamma}\theta\right]} \qquad (67)$$

For C = 1, these inqualities will be the same as (38). Further, both (60) and (64) will be identical with (36). This means that if = 0 at the date of full retirement, it will be having the same value over the whole period. In other words, if we draw the path of domestic income with and without the loan, the two curves will be parallel up to the point h + e.

On the other hand, k = c + e is a limiting point of the present case. Equation (59) applies and if substituted in (61) gives:

$$\theta = C + \frac{C - 1}{e\frac{\sigma}{\gamma}}$$

Substituting this value in (66) we can solve to obtain the same boundaires defined by (54). Notice that to solve for the upper bound, we have to put (e-1) equal to e qpproximately.

(2) <u>$k > e + c$:</u>

The criterion should be based, in this case on (28), which when equated to zero gives:

$$\frac{\sigma}{\gamma}(K - 1) - b\,\frac{K}{CE}(E - 1) = 0$$

Putting,

$$\psi = \frac{K - 1}{K} < 1 \qquad (68)$$

we obtain:

$$b = \frac{\sigma}{\gamma}\psi\left(\frac{CE}{E - 1}\right) \qquad (69)$$

The approximate solution of this formula for σ gives:

$$\sigma = \left(\frac{b}{C\psi} - \frac{1}{e}\right)\gamma \qquad (70)$$

while

$$r = \frac{2\psi C\, e\,\frac{\sigma}{\gamma} + 2\,(C\psi - 1)}{2e - (e - 1)\,C\psi - e\,(e - 1)\,C\psi\frac{\sigma}{\gamma}} \qquad (71)$$

To evaluate the advantage criterion μ, i.e., the increase in the country's capital at the end of the repayment period we substitute (69) in (47) to obtain.

$$\mu = CE\,(1 - \psi) = \frac{CE}{K} < 1 \qquad (72)$$

This means that as soon as the projects financed by the loan are completed the full value of the loan is added to the country's capital. A relatively high annuity leads to a gradual decrease in this additon. At $t = h + e$ only a fraction E^{*}/K remains. But at the subsequent years, a multiplier process brings this fraction up to unity again, and the country would have regained the full capital initiated by the loan at the point $t = f + k$.

Substituting (72) in (51) we obtain the boundaries for e. Using the usual approximations, we can write:

$$\left(\frac{1}{C\psi} - 1\right)\frac{\gamma}{\sigma} \leqslant e \leqslant \left(\frac{2}{C\psi} - 1\right)\frac{\gamma}{\sigma} + \frac{2}{\psi} \qquad (73)$$

If $C \geqslant 1/\psi$, the lower bound of e would be zero. If $C \geqslant 2/\psi$, both bounds will be zero which means that any rate of interest will be permissible whatever the period of repayment.

If $\psi = 1$, equation (72) shows that $\mu = 0$. Formulae (69), (70) and (73) will be the same as (56)-(58), given in case (2) of the previous section. However, this condition cannot be realized unless $1/K = 0$, which means that the period of replacement k is infinite. However, if we denote the annuity in the case $\mu = 0$ by b^*, it can be seen that $b^* = b/\psi$, where b is defined by (69). Hence the value of σ will be the same in both cases, as can be seen from comparing (57) and (70).

If C = 1, the advantage criterion will be $\epsilon = E/K$, which is the same as (I/59). We have to substitute this value in the other formulae. On the other hand, the special case k = c + e leads to the same results obtained for $\mu = 1$ in the previous section.

VIII. Effects of Postponing Repayments:

Our analysis was based so far on the assumption that repayments get first priority on the country's resources. Whatever left after meeting the debt service can be directed towards domestic investments. This need not be the only alternative open, and we might investigate the effects of an approach which might be either explicitly or implicitly adopted.

Suppose that the country plans its investments so as to make use of all resources raised through savings out of its own flow of incomes. This means that when it accepts a given loan at a certain point of time, it will be ready to service it only out of the savings generated as a result of the loan. If it does not get enough resources to face this obligation, it has to borrow the difference, and we assume that it will do that at the same rate of interest. At this stage we assume that it will be doing that on the assumption that the extra finance is a short-term means which is hoped to be repaid in the following year. However, if the savings out of the loan - income fall short of the fixed annuity at any point of time, they will remain to be so over the whole period of repayment. This means that new loans have to be contracted in order to cover the gap and to meet the obligations of the extra debts. In other words, the gap will cause a series of loans which work at a compound rate of interest.

Let us assume that a loan of one unit is contracted in the base year. Then the annuity will be b, while the savings out of the extra income will be $\frac{\sigma}{\gamma}$ over the whole life of the projects thus financed (which we assume to be infinite). If,

$$d = b - \frac{\sigma}{\gamma} > 0 \qquad (74)$$

the country has to pay the full amount b, and borrow the difference d, in year 1. In year 2, there will be another gap d, besides the amount d borrowed in year 1, plus interest on it. The total of these two we call u_2 (u for unpaid). Thus:

$$u_1 = d$$

$$u_2 = d + d\,(1 + r)$$

Again in year 3 a new gap d will be coupled with the full amount of u_2 plus interest on it

$$u_3 = d + u_2\,(1 + r) = d + d\,(1 + r) + d\,(1 + r)^2$$

In general, for any year $t \leqslant e$:

$$u_t = d + u_{t-1}\,(1 + r) = d + d\,(1 + r) + \ldots\ldots + d\,(1 + r)^{t-1}$$

Or,

$$u_t = \frac{d}{r}\left[(1 + r)^t - 1\right] \qquad (t \leqslant e) \qquad (75)$$

In particular

$$u_e = \frac{d}{r}\,(\mathscr{E} - 1), \qquad \mathscr{E} = (1 + r)^e \qquad (76)$$

After that date, repayments of the original debt cease, and the whole amount $\frac{\sigma}{\gamma}$ will be meeting the u's. Therefore, in any year $t = e + i$,

$$u_t = u_e\,(1 + r)^i - \frac{\sigma}{\gamma}\,\frac{1}{r}\left[(1 + r)^i - 1\right]$$
$$= \frac{1}{r}\left[\frac{\sigma}{\gamma} - \left(\frac{\sigma}{\gamma} - ru_e\right)(1 + r)^{t-e}\right] \qquad (77)$$

So long as $u_t > 0$, the process goes on. At some future date, $t = n$, say, it will stop provided that:

$$b_n = (1 + r)\;u_{n-1} \leqslant \frac{\sigma}{\gamma}$$

Substituting from (77) we get the condition that:

$$(1 + r)^{n-e} \geqslant \frac{\sigma}{\gamma} \Big/ \left(\frac{\sigma}{\gamma} - ru_e\right) \qquad (78)$$

In other words:

$$n \geqslant e + \frac{1}{\log (r + r)} \left[\log \frac{\sigma}{\gamma} - \log \left(\frac{\sigma}{\gamma} - ru_e\right)\right] \quad (79)$$

To obtain the actual rate of interest, we use an adaptation of (32), noticing that apart from the last instalment, the annuity is effectively $\frac{\sigma}{\gamma}$, while the repayment period is n.

$$r' = \frac{2 \left[n\frac{\sigma}{\gamma} + \left(b_n - \frac{\sigma}{\gamma}\right) - 1\right]}{n \left[2 - (n - 1)\sigma/\gamma\right]} \quad (80)$$

Starting year n + 1, the country will possess a total capital stock equal to the total value of the loan plus what it could generate without it.

However, to be able to accomplish this process the need for extra borrowing after year e should be diminishing. In other words, we should have:

$$u_t < u_{t-1} \qquad (t > e)$$

Substituting from (77) we find that this means

$$\left(\frac{\sigma}{\gamma} - r\, u_e\right) > 0$$

By (76), this means that

$$b < \frac{\sigma}{\gamma} \left(\frac{\mathcal{E}}{\mathcal{E} - 1}\right) \quad (81)$$

The meaning of this condition is evident: The amount $\frac{\sigma}{\gamma}$ allocated to the debt service should be greater than interest charges on u_e. When this happens, we can make sure that a part of u_e will be rapid in year e + 1 so that the amount remaining will be smaller than u_e and hence interest charges on it will be smaller also.

<u>Example</u>:

Suppose that r = 2.5%, e = 12 years;
$\gamma = 3$. If $\sigma = 0.10$, then $\sigma/\gamma = 0.033\ 333$.
By (3), b = 0.095 238. Hence:

$$d = b - \frac{\sigma}{\gamma} = 0.061\ 905$$

In other words, only one-third of the annuity will he paid, and the rest postponed. By year 12, the amount unpaid will be

$$u_e = 0.061905 \frac{(0.344889)}{0.025} = 0.854014$$

i.e., about 85% of the loan itself.

$$\therefore \quad \frac{\sigma}{\gamma} - r\, u_e = 0.011983$$

This means that the savings out of the loan can cover the interest charges on the remainder after year 12, and there remains a part to pay off a part of that remainder. Using (79) we find that:

$$n \geqslant 12 + \frac{2.5228787 - \bar{2}.0785656}{0.0107239} = 53.43$$

or

$$n = 54$$

We calculate the remainder at the end of year 53:

$$u_{53} = \frac{1}{0.025} \left[0.033333 - 0.011983 \times (1.025)^{41} \right] = 0.014153$$

Hence the last instalment will differ from the others:

$$b_{54} = 0.014153 \times 1.025 = 0.014507$$

The actual rate of interest is:

$$r' = \frac{2 \left[53 \times 0.033333 + 0.014507 - 1.000 \right]}{54 \left[2.000 - 53 \times 0.033333 \right]} = 12.4\%$$

The same process can be repeated for other values of σ , as summarized in the following

σ	0.10	.015	0.20	0.25	0.286
n	54	28	19	15	12
r'	12.4	3.97	3.51	2.74	2.50

The effect of the increase in the value of σ is appreciable, especially on the lower end of the table.

In fact, two critical values of σ can be calculated. The one ensures that by the end of e, the full amount of the loan is repaid.

This value is given by condition (36) ensuring that $\mathcal{E} = 1$. On the other hand we have condition (81) beyond which the loan will never be repaid. The range of variation of σ for a given b will, therefore, be:

$$b\left(\frac{\mathcal{E}-1}{\mathcal{E}}\right)\gamma < \sigma \leqslant b\gamma \qquad (82)$$

In the above example, given that b = 0.095238 and $\mathcal{E}$ = 1.344889, the m. p.s. should fall within the range:

$$0.073359 < \sigma \leqslant 0.285714$$

If σ is exactly at the lower bound, the amount σ/γ will be just sufficient only to pay interest changes on the remainder which means that the country has to postpone it indefinitely.

It should be noticed that if a value of σ within this range is accepted it could be substituted in (82) to obtain the value of $b = \sigma/\gamma$. Substituting in (32) we could obtain the value of e corresponding to the given r. In the present example, if σ = 0.10 the value of e would have been 40.4, or nearly 41 as compared with 54 necessary to account for postponement.

IX - Conclusions:

In this part of our work we have concentrated our attention on a rule of servicing debts which is expected to redistribute the burden over time, more in favour of the debtor country. It was found that this rule has a number of favourable effects:

1- The impact of the annuity is relatively smaller at the beginning of the repayment period, during which income is relatively low.

2- Total outflow whether absolute or capitalized is smaller than in the case of the decreasing annuity, at the same r and e.

3- The lower boundaries on the repayment period (conicidiag with zero interest)are the same as in the decreasing annuity rule, for similar conditions.

4- However, it is possible to find finite values of the repayment period at which any interest rate will be admissible according to any of the advantage criteria contemplated.

5- It follows also that the absolute upper limit on the rate of interest is infinitely higher in the present case.

6- At the same time grace periods and gestation periods have the same alsolute effects on the advantage criteria. As in the decreasing amunity case, the effect of grace periods is quite appreciable, and it enables a considerable reduction in the minimum repayment period required to achieve a certain advantage at a given rate of interest.

7- These findings point in the same direction. If a certain benefit is to be obtained from the loan, concessions at the earlier periods increase the powers of the country for further repayment. Such concessions might take the form of full grace periods or at least lower annuities at the earlier periods, relative to the country's income.

8- We have also investigated the effects of partial postponement of the annuities. Critical values for the m.p.s. were obtained, below which such a postponement will go on indefinitely and at the same time the actual rate of interest will become infinitely large.

9- This last remark shows that an attempt at rescheduling debt is not necessarily a beneficial act (abstracting from the transfer problem proper). In the example given above, the acceptance of a given rate of interest leads to the direct negotiation of a period of repayment of 41 instead of the 54 actually necessary to pay off the same debt.

The key role played by the m.p.s. in connection with the servicing of debts, and the effects of rescheduling need further study. We shall deal with these problems in the following parts, paying some attention to the work done by the I.B.R.D. experts.