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Solution Methods For Integer Programming Problems

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Contents

Introduction:

Integer programming is an extension of linear programming, that is to say we begin by forgetting the integer requirement and solving the problem. If the solution has all variables of the integer form, we have found an optimum, integer solution. If it dosnot we continue by adding new constraint, which is called a cut. This cut permits the new set of the feasible solutions to include all feasible integer solutions for the origind constraints, but it dose not include the optimal non integer solution originally found.

In production process as also for economic planning one use integer constraints for known variables also for no exact solvalble algorithm at hand.

The aim of this paper is to give a theoritical as also a practical studies for the integer programming techniques. In the first part of this paper some approaches to integer programming problems are mentioned. Also some remarks on the property of each technique were given with comparison on these properties.

A consideration of the types of the cut due to Gomory, Dantzig. and Land and Doig, and the solution mehtods due to them are illustrated.

At the end of this parper there is a comparision between those methods showing it advandage and disadvantage, also examples are formulated ans solved for each technique and in the comparision case.

1. Integer Programming:

1-1. GOMORY METHODS:

The following is a method for getting the optimum of on integer linear programming problem, this method was developed by Gomory*.

The problem is to get a non-negative values X_j which maximize the function

$$\eta = \sum_{j=1}^{n} a^1_{o_j} X_j \ldots \quad \max \qquad (1)$$

under the constraints

$$\sum_{j=1}^{n} a_{ij} X_j \leqslant a_{io} \qquad (2)$$

(i = 1, 2, ... m)

$$X_j > 0 \qquad (3)$$

Using the simplex method we can get an optimum with non-integer values for the above problem. In solving such problem we introduce slack variables $\bar{X}_i$, which are considered as basic variables at the first step, beside we take η as basic variable i.e all basic variables eill be $(\bar{X}_i, \eta)$ so relation (1) and (2) will be in the form.

* Gomory, R.E, Baumal, W., J,: Integer programming and pricing, Econometerica Bd. 28, 1979.

$$\eta = a_{oo} + \sum_{j=1}^{n} a_{o_j} \; (-X_j) \qquad (4)$$

$$a_{oo} = 0$$

$$a^{1}_{o_j} = - a_{oj}$$

$$\bar{X}_i = a_{io} + \sum_{j=1}^{n} a_{ij} \; (- X_j) \qquad (5)$$

$$(i = 1, \ldots m)$$

Relations (4) and (5) gives the basic table of the simplex method, which can be written in the following Matrix form.

$$X^{0}_{B} = A^{o} X^{0}_{N}$$

The vector X^0_B have m+1 elements. η and the basic variables.

X^{o}_{N} includes the non-basic variables $(- X^{N}_{j})$ and a 1 as first element. A^{o} represents all constant

$$o = \begin{pmatrix} a^{o}_{oo} & & a^{o}_{on} \\ \cdot & & \\ \cdot & & \\ \cdot & & \\ a^{o}_{mo} & & a^{o}_{m} \end{pmatrix}$$

The simplex method make interchanges between the basic and non-basic variables and give the new equation

$$X_B^K = A^K X_N^K \qquad (6)$$

The solution of the linear programming problem is found when.

$$a_{oj}^K \geq 0 \qquad (7) \qquad (j = 1, 2, \ldots, n)$$

$$a_{io}^K \geq 0 \qquad (8) \qquad (i = 1, \ldots\ldots m)$$

Relation (7) gives the optimal criteria and (8) is the condition of the non-negativity a (3). The non- integer solution is then obtained from the column of A^k

$$X_B^k = \begin{bmatrix} \eta \\ x_1^B \\ \\ \\ \\ x_m^B \end{bmatrix} = \begin{bmatrix} a_{oo}^k \\ a_{10}^k \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ a_{mo}^k \end{bmatrix} + \begin{bmatrix} a_{01}^k \ldots & a_{on}^k \\ a_{11}^k \ldots & a_{1n}^k \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ a_{m1}^k \ldots & a_{mn}^k \end{bmatrix} \begin{bmatrix} -x_1^N \\ -x_2^N \\ \\ \\ \\ -x_n^N \end{bmatrix}$$

1-2. ADDITION INEQUALITIES

The simplex method transform the matrix A^o in the form (7) and (8). The matrix A^k is not in general sufficient condition for integerablity, for this reason we must consider for the non-negative integer solution (X_B^1) that

$$X_B^1 \equiv 0 \quad (\text{mod } 1) \tag{9}$$

This means that the vectors X_B^1 and zero are different through integers. Using (9) and (6) for the non-basic variables in the integer solution (X_N^1) then

$$0 = A^K X_N^1 \tag{10}$$

This system must be fulfilled when the non-basic variables are integer basic variables. From (10) we can bild as system of equations, for example

$$0 \equiv ao + \sum_{j=1}^{n} a_j (- X_j^N)^* \tag{11}$$

If we take from (11) that for all $j \geq 1$, $a_i \geq 0$ then we can write

$$a_o = \sum_{j=1}^{n} a_j X_j^{N1}$$

The right hand side of this equation have only non-negative value different from that of the left-hand side with an integer number. Separeting the left side in an integer number (n_0) and a non-negative rest part as f_0 with

$$n_0 = o$$

$$o < f_0 < 1$$

and subtracting a_o from the left side, then can the difference between the left side and the right side be fo, 1+fo, 2+fo, ..., and at any case we have

$$f_0 \leq \sum_{j=1}^{n} a_j \, x_j^{N1} \qquad (12)$$

Relation 12 can be written in the form

$$f_0 \leq \sum_{j=1}^{n} f_j \, x_j^{N1} \qquad (13)$$

Introducing a slack variable (S) we get

$$a = - f_0 - \sum_{j=1}^{n} f_j \, (- x_j^{N1}) \qquad 14)$$

$$a \geq o$$

If under the value are $a_j < o$, then we can through addition of non-basic variable bring a_j to $f_j > o$.

1-3. THE DUAL SIMPLEX - METHOD TO GET FEASIBLE SOLUTION:

Now we begin with the non integer solution (9) with $X_N = (1, 0, 0, \ldots)$:

$$X_B = A \, X_N$$

Is X_B not integer, so some of the

$x_1^B, 0, x_n^B = a_{10} \ldots a_{mo}$ from $n_1 + f_{10} \ldots n_m + f_{m0}$ are with n integer and $0 < f < 1$. Putting some of these value in (14), we get.

$$a = - f_0 .$$

From the non-negativity condition is the solution unfeasible. i.e the non-integer solution after this boundes is not feasible due to relation (14). For this reason we deal with the dual method.

We begin now with the simplex table with

$$X_B = A\ X\ N$$

in table form

	1	$-X_1^N$ -----	X_n^N
=	a_{00}	a_{01}	a_{0n}
X_1^B =	a_{10}	a_{11}	a_{1n}
⋮			
X_m^B	a_{m0}	a_{m1}	a_{mn}

The table is primal feasible when all

$$a_{10} \geq 0 \quad (i = 1, \dots m)$$

We take now the most negative values under $a_{0j} < 0$ and define the variables, in which the raw (r) was selected as a new basic variable.

The new basic variables will be through

$$\min_{i} \frac{a_{io}}{a_{ir}} \qquad (i = 1, \dots m) \quad (a_{ir} > 0)$$

The table is then dual feasible, when all $a_{oj} > 0$ ($j = 1, \dots n$). Now select the values $a_{io} \geq 0$ ($i = 1, \dots, m$) the most negative, and define the variable X_u^B in the column in which this element is as new nonbasic-variable the new basic variable will be

$$\max_{j} \frac{a_{oj}}{a_{uj}} \qquad (j = 1, \dots n) \quad (a_{oj} < o)$$

The table is then primal and also dual feasible. So we can get the optimal one.

Since this method of Gomory does not take in consideration equation (14), all a_{oj} are non-negative and one element of a_{io} is negative ($-f_0$) then the dual simplex algorithm is the only solution method

1-4 Selection of effective inequality:

In order to get smaller number of iterations we must develop certain criteria for selection of the variables i.e. inequalities (13) must be reduced, for this reason we can called D as the product of all given main elements.

$$\prod_k . \; P_k = D$$

and then writting all number in the table in the form H/D, 30 are the value of H always integer numbers and we get an inequality

$$f_o^o \leq \sum_{j=1}^{n} f_j^0 \; X_j^N \qquad (15)$$

which help in the selection rule i.e that with the greater value of $f_0^0 < 1$ to be defined. Now we write

$$f_0^0 = \frac{ho}{D}$$

which gives (h_0, D) the greatest value.

From the Euklidion algorithms (ho, D) as integer linear combination has the form

$$(h_0, D) = m_o D + n_0 h_0.$$

Now multiplying $f_0^0 = \frac{h_0}{D}$ with no we get

$$n_0 f_0^0 = f_0^1 = \frac{(h_0, D)}{D}$$

this f_0^1 is the smallest value of f_0 on the other hand if we multiply by $(D - n_0)$ we get

$$(D - n_0) \quad f_0^0 = f_0^2 = 1 - \frac{(h_0, D)}{D}$$

which gives the greatest f_0.

The selection of the greatest (f_0) is the same as the selection of the least negative value in the primal problem but hier is more easy.

A second criteria for selection can be used, this criteria depends on the boundes of the polyeder of the non-basic variables, which can be bring to strong small value, and then the objective function reaches its values. To do this we begin by selecting the non-basic variables with the minimum a_{0j} in the column, $\min a_{0j} = a_{0k}$

The value f_k^0 from (15) can be written in the form

$$f_k^0 = \frac{h_k}{D}$$

then

$$\frac{f_k^0 \quad D}{(h_k, D)} = 0$$

with

$$\frac{f_0^0 \quad D}{(h_k, D)} \not\equiv 0$$

Now the ratio

$$\frac{f_0^1}{f_k^1} = \frac{f_0^1}{0} = \infty$$

is maximum

we get now the selected inequality through multiplication with $\frac{D}{(h_k, D)}$

in the form

$$\frac{f_0^0 \quad D}{(h_k, D)} = 0$$

and the maximum ratio we get through the multiplication with n_k where n_k

is given with

$$(h_k, D) = m_k D + n_k h_k$$

This selection criteria maximize in all case the relation.

$$\max_{r} \frac{f_0^r}{f_k^r} = \frac{f_0^u}{f_k^h} \quad (r = 1, \ldots u, \ldots D)$$

which also maximize the objective function

$$\Delta \eta = f_0^u \left(\min_j \frac{a_{0j}}{f_j^u}\right)$$

It is met only when we have a starting selected value as

$$\min_j \quad a_{0j} = a_{0k}$$

at the same time the minimum of the ratio

$$\min_j \quad \frac{a_{0j}}{f_j^u} = \frac{a_0^k}{f_k^u}$$

then we get

$$\max \Delta\eta = \max_r \left[f_o^r \min_j \left(\frac{a_{0j}}{f_j^r}\right)\right] = f_0^u \frac{a_{0k}}{f_k^u}$$

For this selection criteria we see that:

a) it is applicable for smaller D-values and it of simpel application.

b) It take smaller number of calculations.

c) by greater.D. values we need additional calculation to reach the second selection criteria, which helps to define the inequality with

$$\max_r \quad f_0^r.$$

d) with selection of inequalities we can change the selection rule from table to table.

e) new tables with $a_{io} \geq 0$ are determind by the dual method which is directed related to the slack variables of the new restirction in the basis.

f) it we can also find for the integer solution an integer table this means all elements of the simplex table is brough to integer values.

g) Since the a_{i0} column in an Integer solution then only $f_0 = 0$ included, the change of the objective function is equal to zero

$$\Delta \eta = (f_0^u = 0)\ (\min_j \frac{a_{0j}}{f_j^u}).$$

1-5. Exampel and geometeric interpretion of the method:

It is asked to determine the values X_1 and X_2 that maximize the function

$$\eta = 5X_1 + 6X_2$$

under the constraints

$$4X_1 + X_2 \leq 32$$
$$5X_1 + 2X_2 \leq 26$$
$$6X_1 + 8X_2 \leq 67$$
$$6X_1 + 12X_2 \leq 89$$

X_1 and X_2 are non-negative.

The slack variables $\bar{X}_1, \ldots, \bar{X}_4$ and the simpelx tabels (0, A, B, C, D) gives us the non-integer solution.

$x_1 = 6\frac{1}{6}$, $\bar{x}_1 = 3\frac{7}{12}$ $\bar{x}_2 = \bar{x}_3 = 0$

$x_2 = 3\frac{3}{4}$, $\bar{x}_4 = 7$ $\eta = 53\frac{1}{3}$

0	1	$-x_1$	$-x_2$
$\eta =$	0	-5	-6
$\bar{x}_1 =$	32	4	1
$\bar{x}_2 =$	26	3	2
$\bar{x}_3 =$	67	6	8
$\bar{x}_4 =$	89	6	[12]

A	1	$-x_1$	$-\bar{x}_4$
η	$44\frac{1}{2}$	-2	1/2
$\bar{x}_1 =$	295/12	7/2	$-\frac{1}{12}$
$\bar{x}_2 =$	67/6	2	-1/6
$\bar{x}_3 =$	23/3	[2]	-2/3
$x_2 =$	89/12	$\frac{1}{2}$	1/12

B	1	$-\bar{x}_3$	$-\bar{x}_4$
$\eta =$	321/6	1	-1/6
$\bar{x}_1 =$	134/12	-7/4	13/12
$\bar{x}_2 =$	21/6	-1	[½]
$x_1 =$	23/6	½	$-\frac{2}{6}$
$x_2 =$	66/12	$-\frac{1}{4}$	$\frac{1}{4}$

C	1	$-\bar{x}_3$	$-\bar{x}_2$
$\eta =$	531/3	2/3	1/3
$\bar{x}_1 =$	43/12	5/12	13/6
$\bar{x}_4 =$	7	-2	2
$x_1 =$	37/6	$-\frac{1}{6}$	2/3
$x_2 =$	15/4	$\frac{1}{4}$	- ½

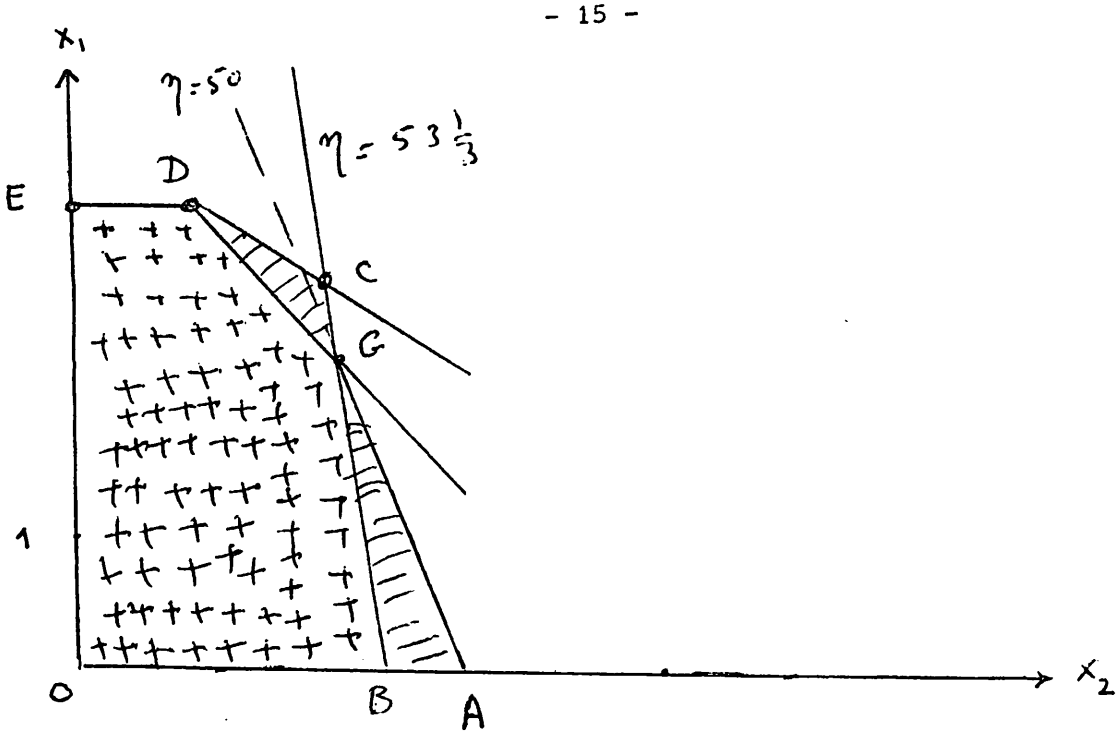
x1
η = 50
η = 53 1/3
D
E
C
G
1
0
B
A
x2

Fig (1)

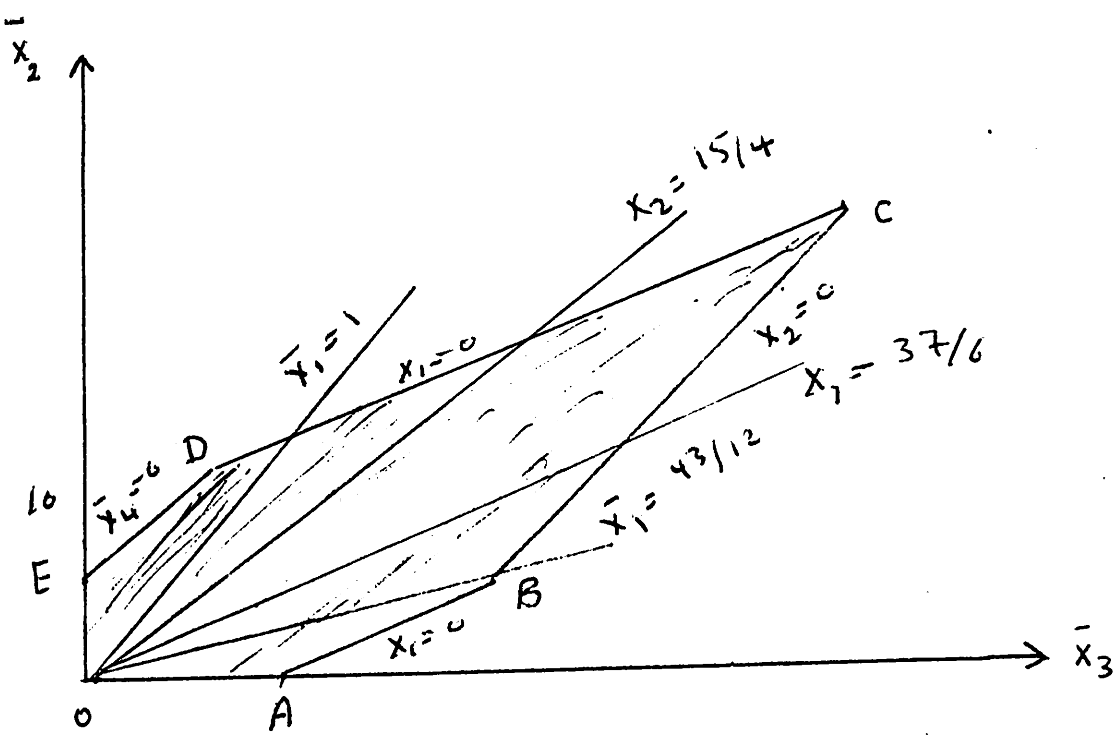
x̄2
x2 = 15/4
C
x̄1 = 1
x1 = 0
x2 = 0
X1 = 37/6
x̄4 = 0
D
10
E
x̄1 = 43/12
B
x1 = 0
x̄3
0
A
10

Fig (2)

In fig. (1) we see the normal graphical solution of the problem (non-integer) which gives the optimal point ($53\frac{1}{3}$) and $X_1 = 6\frac{1}{6}$, $X_2 = 3\frac{3}{4}$

In Fig.(2) is non-integer solution table in which $\bar{X}_2 - \bar{X}_3$ are represented*.

The bounded polyeder is formed by the basic variables (X_1, X_2, $\bar{X}_1$, $\bar{X}_4$) in which the point $\bar{X}_2 = 0$ and $\bar{X}_3 = 0$ (non-basic variable

From table C we see that the smallest general index is $D = \prod P_k = 12.2\frac{1}{2} = 12$

so the only equation which is smaller than 12 (index) is

$$X_1 = 3\frac{7}{12} + 5/12\,(-\bar{X}_3) - 2\frac{2}{12}\,(-\bar{X}_2)$$

Seperating the non-integer value of this equation into (n) and (f) values, then we get after that the factor - 2 2/12 through addition an integer postive value.

$$f_0 = \frac{7}{12},\ f_3 = 5/12,\ f_2 = \frac{10}{12}$$

(15) gives then

$$\frac{7}{12} < \frac{5}{12}\bar{X}_3 + \frac{10}{12}\bar{X}_2 \qquad (16)$$

or after introducing a slack variable we get

$$a_1^1 = -\frac{7}{12} + \frac{5}{12}\bar{X}_3 + \frac{10}{12}\bar{X}_2 \qquad (17)$$

which is a resteraction to be considered in the optimal table.

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* See Tucker, A.W: Gomory's Algorithms for integer programs Memorandum for social - okomouisk oslo-1969.

As an addition row we have

	1	$-x_3$	$-x_2$
$a_1 =$	$-7/12$	$-5/12$	$-\frac{10}{12}$

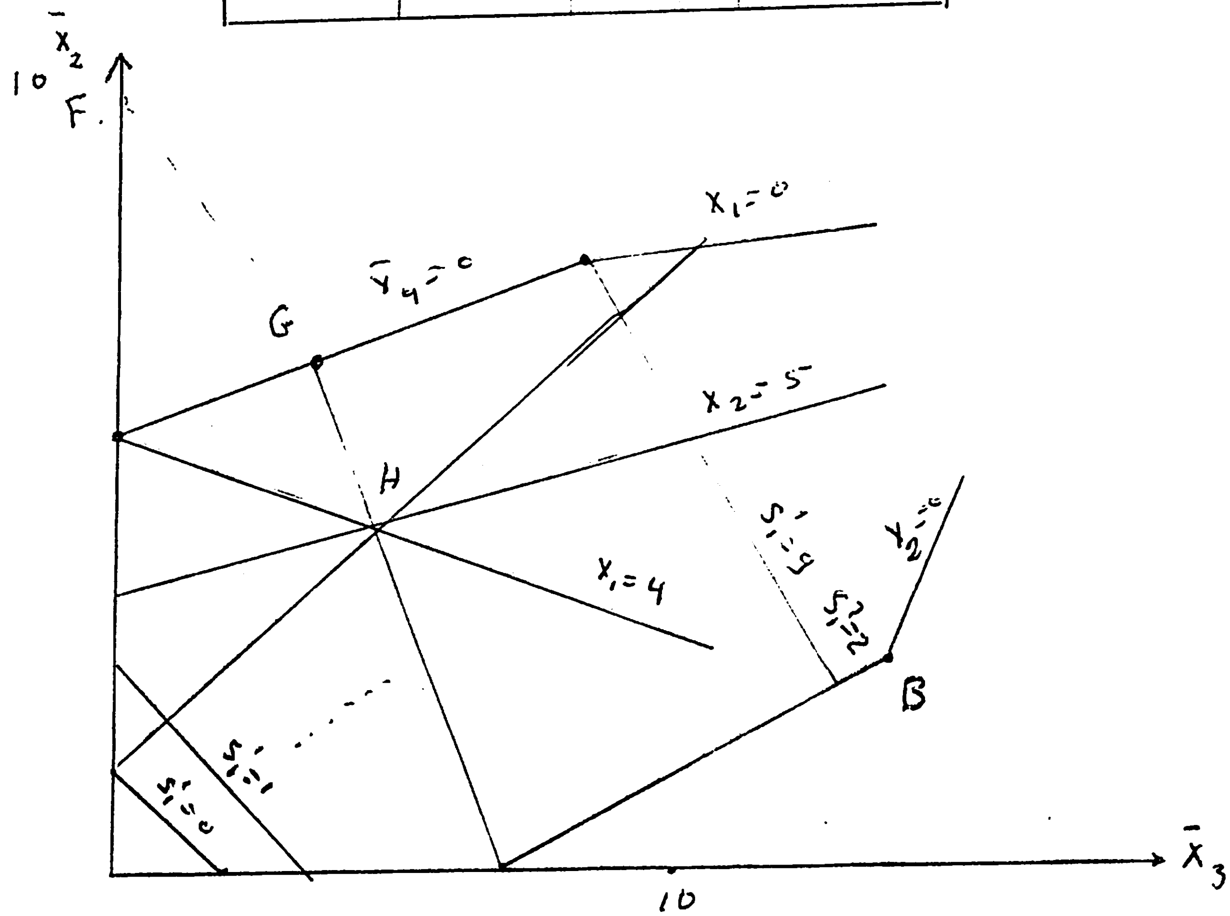


fig. (3)

Fig (B) gives a part of fig (2). The bounds (17) is for the values a_1^1 = 0, 1, 2, 3 ..., 9. The integer values of the basic variables can be as parallel lines to the bounded polyhedral (see fig (2)).

Now we are able as before that from restraction (17) we derive a number of resteractions. We select now the maximum f_0. The common divisior row is D = 12 and $h_0 = 7$

i.e.

$$(h_0, D) = (7, 12) = 1$$

the integer linear combination is

$$mD + n\, h_0 = -4.12 + 7.7 = 1$$

with n = 7. Multiplying (17) by

$$(D - n) = 5 \text{ we get}$$

$$2\,\frac{11}{12} < 2\,\frac{1}{12}\,\bar{X}_3\; 4\,\frac{2}{12}\,\bar{X}_2 \qquad (18)$$

or in row weise of the simplex table after introducing the slack variables we can write

$$a_1^2 = -\frac{11}{12} - \frac{1}{12}(-\bar{X}_3) - \frac{2}{12}(-\bar{X}_2) \qquad (19)$$

This resterication is obtained from the first selective criteria.

The inequality (19) with the values S_1^2 = 0, 1, ... are obvious with the inequality (17) with the values a_1^1 = 4, 9, ... (see fig 3).

Since only branches with $S \geq 0$ are taken. This mean that the boundes are brough to a small branches. which mean that the calculations are less in steps.

The second selective criteria is to choose the column with the minimum a_{0j}, which is colomum $\bar{X}_2$ $(a_{01} = \frac{2}{3}, a_{02} = \frac{1}{3})$. For f_2 also is the general part

$$(h_2, D) = (10, 12) = 2$$

to form.

now since

$$\frac{f_0 \; D}{(h_2, D)} = \frac{7.12}{12.2} \neq 0$$

then we must multiply (16) with $D/ (h_2, D) = \frac{12}{2}$ then we get

$$3 \frac{6}{12} < 2 \frac{6}{12} \bar{X}_3 + S \bar{X}_2 \; .... \qquad (20)$$

and after introducing the slack variable we get

$$a_1^3 = \frac{6}{12} - \frac{6}{12} (- \bar{\bar{X}}_2) \qquad (21)$$

Now we define all the groups of reduced restrications which is given through multiplying (16) with the factors 1, 2, ... 12. The resteraction is not written but only gives values of f_0-, f_3- and f_2.

we get then the following table.

		f_0	f_3	f_2
1.	1/12	(7	5	10)
2.	1/12	(2	10	8)
3.	1/12	(9	3	6)
4.	1/12	(4	8	4)
5.	1/12	(11	1	2)
6.	1/12	(6	6	0)
7.	1/12	(1	11	10)
8.	1/12	(4	8	4)
9.	1/12	(3	9	6)
10.	1/12	(10	2	4)
11.	1/12	(5	7	2)
12.	1/12	(0	0	0)

The minimum of a_{0j}/f_j for $a_{01} = 2/3$, $a_{02} = \frac{1}{3}$ and for $f_j > 0$ are shown.The greatest value of the objective function is with the fifth restrication.

$$\Delta \eta = f_0 \left(\min \frac{a_{0j}}{f_j}\right) = \frac{11}{12} \cdot \frac{1}{3} \cdot \frac{2}{12} = \frac{22}{12}$$

The restrication is an effective restrication taking this restrication in table (e), so we get the following table (C) and from which we get table (F). In table F we have a solution, but the dual not yet feasible (point F) in fig (1) and (3). Table G represents feasible dual solution. The new resterication is

$$5/6 < \frac{1}{6} f_4$$

$$S_2 = - 5/6 - \frac{1}{6} (- \bar{X}_4) \qquad (22)$$

C^1	1	$-\bar{X}_3$	$-\bar{X}_2$
=	$53\frac{1}{3}$	2/3	$\frac{1}{3}$
$\bar{X}_1$ =	43/12	5/12	26/12
$\bar{X}_4$ =	7	-2	2
X_1 =	37/6	$\frac{1}{6}$	2/3
X_2 =	45/12	$\frac{1}{4}$	$-\frac{1}{2}$
S_1 =	11/12	$\frac{1}{12}$	2/12

F	1	$-\bar{X}_3$	$-S_1$
	$51\frac{1}{2}$	$\frac{1}{2}$	2
$\bar{X}_1$ =	31/2	3/2	-13
$\bar{X}_4$ =	-4	-3	12
X_1 =	5/2	$-\frac{1}{2}$	4
X_2 =	13/2	$\frac{1}{2}$	-3
X_2	11/2	$\frac{1}{2}$	-6

G	1	$-\bar{X}_4$	S_1
$\eta =$	505/6	1/6	0
$\bar{X}_1 =$	27/2	1/2	-7
$\bar{X}_3 =$	4/3	$\frac{1}{3}$	-4
$X_1 =$	19/6	-1/6	2
$X_2 =$	35/6	1/6	-1
$\bar{X}_2 =$	29/6	1/6	-4
$S_2 =$	-5/6	1/6	0

H	1	$-S_2$	$-S_1$
$\eta =$	50	1	4
$\bar{X}_1 =$	11	3	-7
$\bar{X}_3 =$	3	-2	-4
$X_1 =$	4	-1	2
$X_2 =$	5	1	-1
$\bar{X}_2 =$	4	1	-4
$\bar{X}_4 =$	5	-6	0

The solution is now

$X_1 = 4$ $\qquad X_1 = 11$ $\qquad X_4 = S$

$X_2 = 5$ $\qquad X_2 = 4$ $\qquad \eta = 50$

$X_3 = 3$

Both resterication (19) and (22) are in fig (1) in the X_1-X_2 plan.

In order to get the equation of the additional constraint in the first table we write $\bar{X}_2$, $\bar{X}_3$ and $\bar{X}_4$

as

$$\bar{X}_2 = 26 - 3X_1 - 2X_2$$

$$\bar{X}_3 = 67 - 6X_1 - 8X_2$$

$$\bar{X}_4 = 89 - 6X - 12\, X_2$$

Substituting these values in (19) and (22) we get

$$a_1 = -\frac{11}{12} + \frac{1}{12}(67 - 6X_1 - 8X_2) + \frac{2}{12})26 - 3X - 2X_2)$$

$$a_2 = 5/6 + \frac{1}{6}(89 - 6X_1 - 12X_2)$$

which gives

$$S_1 = 9 - X_1 - X_2$$

$$S_2 = 14 - X_1 - 2X_2$$

from which we get the additional Integer condition with

$$S_1 + X_2 \leq 9$$

$$X_1 + 2X_2 \leq 14$$

2. Dantzig's Method:

2-1. The Resterations of Dantzig:

Dantzig consider the constraints of a linear programming problem as

$$\sum_{j=1}^{n} a_{ij} \; X_j \geq a^1_{i0} \qquad (26)$$

$$(i = 1, 2, \ldots m)$$

It is asked now about on Extrem point of the convex polyeder which is a solution of a linear objective function. This means that for an Extrem point in the normal case for the n variables in the m inequalties (26) we have

$$\sum_{j=1}^{n} a_{ij}^{1} x_j = a_{i0}^{1} \qquad (27)$$

$$(i = 1, \ldots m)$$

(27) is called Dantzig corner point*. If (27) are only equations then the non-integer solution is obtained through

The additional constraints

$$\sum_{j=1}^{n} a_j^1 x_j = a_0^1 + j' \qquad (28)$$

i.e

$$\sum_{j=1}^{n} a_j^1 x_j = a_0^1 + \alpha \qquad \alpha \geq 1 \quad (28')$$

If a_j^1 and a_0^1 are integer numbers. Then must the inequalaties $^{(28)}$ are common point. for this reason we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{1} x_j = \sum_{i=1}^{n} a_{i0}^{1} + \alpha \qquad (29)$$

$$(\alpha \geq 1)$$

when (27) are n equation, the condition (29) can be written as

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{1} x_j - \alpha = \sum_{i} a_{i0}^{1} \qquad (29')$$

$$\alpha \geq 1$$

For the intial form of (26), in which the slack variables equal to zero if we introduce slack variables, then we get

* Dantzig, G.B: Note on linear programming santo Manica, Calif. Rand Corporation.

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^{1} x_j - \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} a_{i0}^{1} \qquad (26')$$

from (26') and (29') we get

$$\sum_{i=1}^{n} \bar{x}_i = \alpha \qquad \alpha \geq 1 \qquad (30)$$

or

$$\sum_{i=1}^{n} \bar{x}_i \geq 1 \qquad (30')$$

The conditions (30), (30') is the same as restriction (14), Both are not fulfilled in the non-integer optimal part with $\alpha = 1$

Instead of (26) we use the following restriction

$$\sum_{j=1}^{n} a_{ij} x_j \leq a_{i0} \qquad (31)$$

$$(i = 1, \ldots m)$$

Then we have $a_{ij} = - a_{ij}^{1}$ and $a_{i0} = - a_{i0}^{1}$

which transforms (29) & (29') to the form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} (- a_{ij}^{1}) x_j = \sum_{i=1}^{n} (- a_{i0}) - \alpha$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_j = \sum_{i=1}^{n} a_{i0} - \alpha \qquad (32)$$

$$(\alpha \geq 1)$$

Subtracting from (32) the n restrication in which the slack is the Extrem points equal to zero we get

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \ x_j \ + \alpha \ = \ \sum_{i=1}^{n} a_{i0}$$

$$- \ (\sum_{i=1}^{n} \ \sum_{j=1}^{n} \ a_{ij} \ x_j \ + \ \sum_{i=1}^{n} \bar{x}_j \ = \ \sum_{i=1}^{n} a_{i0})$$

$$\sum_{i=1}^{n} \bar{x}_i \ = \ \alpha \qquad \alpha \geq 1 \qquad (30')$$

$$\sum_{i=1}^{n} \bar{x}_i \ \geq \ 1$$

i.e in new feasible optimal table, restriction (30) is fulfilled, it is always

$$\sum_{i} \bar{x}_i \ = \ 1 \qquad (30'')$$

also

$$\sum_{i} x_i \ > \ 1$$

gives a smaller value of the objective function.

2.2 Exampel of the method

Taking the same exampel in the pervious method which is:

The objective function

$$\eta = 5\,X_1 + 6X_2 \ldots \quad \max$$

under the constraints

$$4X_1 + X_2 \leqslant 32$$

$$3X_1 + 2X_2 \leqslant 26$$

$$6X_1 + 8X_2 \leqslant 67$$

$$6X_1 \div 12X_2 \leqslant 89$$

$X_1, X_2 > 0$ and integers.

The non-integer solution is

$$X_1 = 6\tfrac{1}{6} \qquad X_1 = 3\tfrac{1}{12}$$

$$X_2 = 3\tfrac{3}{4} \qquad X_4 = 7 \qquad \eta = 53\tfrac{1}{3}$$

$$X_2 = X_3 = 0$$

The additional restrication (30') is

$$X_2 + X_3 \geqslant 1 \qquad (33)$$

we derive restrication (33) direct from the given constraints. So we add the second and the third constraints with the slack variables $\bar{X}_2, \bar{X}_3 = 0$ and then we get the same singus relation (32) is now in the form.

$$9X_1 + 10\,X_2 \leqslant 93 - 1 \qquad (34)$$

Taking X_1, X_2 as non-basic-variables of the Extem points $(\bar{X}_2, \bar{X}_3)$ from the relations

$$\bar{X}_2 = 26 - 3X_1 - 2X_2$$

$$\bar{X}_3 = 67 - 6X_1 - 8X_2$$

then (34) will be

$$9 \left(\frac{37}{6} - \frac{2}{3} \bar{X}_2 - \frac{1}{6} \bar{X}_3\right) + 10 \left(\frac{15}{4} + \frac{1}{2} \bar{X}_2 - \frac{1}{4} \bar{X}_3\right)$$

$$= 93 - 1$$

i.e.

$$- \bar{X}_2 - \bar{X}_3 \leq - 1 \qquad (35)$$

for integer optimal table we get

also the condition

$$S_1 = -1 -1 \, (- \bar{X}_2) - 1 \, (-\bar{X}_3)$$

Now from table C' we get the set of following tables D, E, F, G, H, J, K and L

c^1	1	$-\overline{x}_3$	$\overline{x}_2$
η =	$53\frac{1}{3}$	2/3	1/3
$\overline{x}_1$ =	43/12	5/12	-13/6
$\overline{x}_4$ =	7	-2	2
x_1 =	37/6	-1/6	2/3
x_2 =	15/4	1/4	-1/2
s_1 =	-1	-1	[-1]

D	1	$-\overline{x}_3$	$-s_1$
η =	53	1/3	1/3
$\overline{x}_1$ =	69/12	31/12	-13/6
$\overline{x}_4$ =	5	-4	-2
x_1 =	33/6	-5/6	2/3
x_2 =	17/4	3/4	$-\frac{1}{2}$
$\overline{x}_2$ =	1	1	-1
s_2 =	-1	[-1]	-1

E	1	$-s_2$	$-s_1$
η =	322/3	1/3	0
$\overline{x}_1$ =	38/12	31/12	-57
$\overline{x}_4$ =	9	-4	6
x_1 =	38/6	-5/6	9/6
x_2 =	14/4	3/4	-5/4
$\overline{x}_2$ =	0	1	-2
$\overline{x}_3$ =	1	-1	1
s_3 =	-1	-1	[-1]

F	1	$-s_2$	$-s_3$
$\eta =$	522/3	1/3	0
$x_1 =$	95/12	88/12	$-\frac{57}{12}$
$x_4 =$	3	-10	6
$x_1 =$	29/6	$-\frac{14}{6}$	9/6
$x_2 =$	19/4	2	-5/4
$x_2 =$	2	3	-2
$x_3 =$	0	-2	1
$s_4 =$	-1	-1	-1

G	1	$-s_2$	$-s_4$
$\eta =$	522/3	1/3	0
$x_1 =$	152/12	145/12	-57/12
$x_4 =$	-3	-16	6
$x_1 =$	20/6	$-\frac{23}{6}$	9/6
$x_2 =$	6	13/4	-5/4
$x_2 =$	4	5	-2
$x_3 =$	-1	-3	1

H	1	$-\bar{x}_3$	$-s_4$
$\eta =$	525/9	1/9	1/9
$\bar{x}_1 =$	34/36	-26/36	-26/36
$\bar{x}_4 =$	7/3	-16/3	- 2/3
$x_1 =$	83/18	-23/18	4/18
$x_2 =$	59/12	13/12	-2/12
$\bar{x}_2 =$	7/3	5/3	-1/3
$s_5 =$	-1	-1	-1

J	1	$-\bar{x}_3$	$-s_5$
$\eta =$	524/9	0	1/9
$\bar{x}_1 =$	337/36	171/36	26/36
$\bar{x}_4 =$	3	- 14/3	-2/3
$x_1 =$	79/18	27/18	4/18
$x_2 =$	61/12	15/12	-2/12
$\bar{x}_2 =$	8/3	2	-1/3
$s_6 =$	-1	-1	-1

B	1	$-s_6$	$-s_5$
η =	524/9	0	1/9
$\tilde{x}_1$ =	166/36	171/36	-197/36
$\bar{x}_4$ =	23/3	-14/3	4
x_1 =	106/18	-27/18	31/18
x_2 =	46/12	15/12	-17/12
$\bar{x}_2$ =	2/3	2	- 7/3
$\bar{x}_3$ =	1	-1	1
s_7 =	-1	-1	-1

L	1	$-s_7$	$-s_5$
η =	524/9	0	1/9
$\tilde{x}_1$ =	-5/36	17/36	-368/36
$\bar{x}_4$ =	37/3	-14/3	26/3
x_1 =	133/18	-27/18	58/18
x_2 =	31/12	15/12	-32/12
$\bar{x}_2$ =	- 4/3	2	- 32/3
$\bar{x}_3$ =	2	-1	2

Which means that the value of η is takes smaller value and iterations methods is getting move and more without getting on integer solution also after 150 iteration steps no optimal integer solution was obtained. This means that the additional conditions put by Dantzig is not helpfull for getting an integer optimal solution for problems in such forms.

3- LAND and DOIG Method

3-1 Method interpretation

The method *consider the optimal solution of linear programming problem without integer conditions. The greatest value of the objective function (η) must be syetematic from the value $\eta^k, \eta^{k-1} \ldots \eta^1, \eta^0$, η^o, in which the variables of the integerablity are introduced.

The problem is in its general forms as

$$\eta = \sum_{j=1}^{n} a_{oj}\, x_j \;\ldots\ldots\ldots\; \max \qquad (36)$$

and the boundes are given thnough

$$\sum_{j=1}^{n} a_{ij}\, x_j \leqslant a_{io} \quad (i=1,2,\ \ldots\ m) \qquad (37)$$

$$x_j \geqslant 0 \qquad (38)$$

The boundes form a convex bounded area. For given values of η can the variables x_j max or min.

Dealing with the variable x_k (j=1, ..., k, .. m) then we get the min or max from the following linear programm.

$$x_k \;\ldots\;\ldots\;\ldots\; \min \qquad (39)$$

$$a_{oj}\, x_j - \eta = 0 \qquad (36)$$

$$a_{ij}\, x_j \leqslant a_{io} \qquad (37)$$

$$x_j \geqslant 0 \qquad (38)$$

This problem can be in the x_k plan represented Fig (4). Since a linear progection of a canvex boundes is convex, then the system (36"), (37), (37), (38), (39) gives also convex bound. In consider-

*Land, A.H., Doig, A.G.: An outomatic method of solwing discrete programing problem. Econometrics Bd 28-1960.

ing of the x_k – axis is the upper bound concave and the lower limits convex (OABCDE in Fig (4)

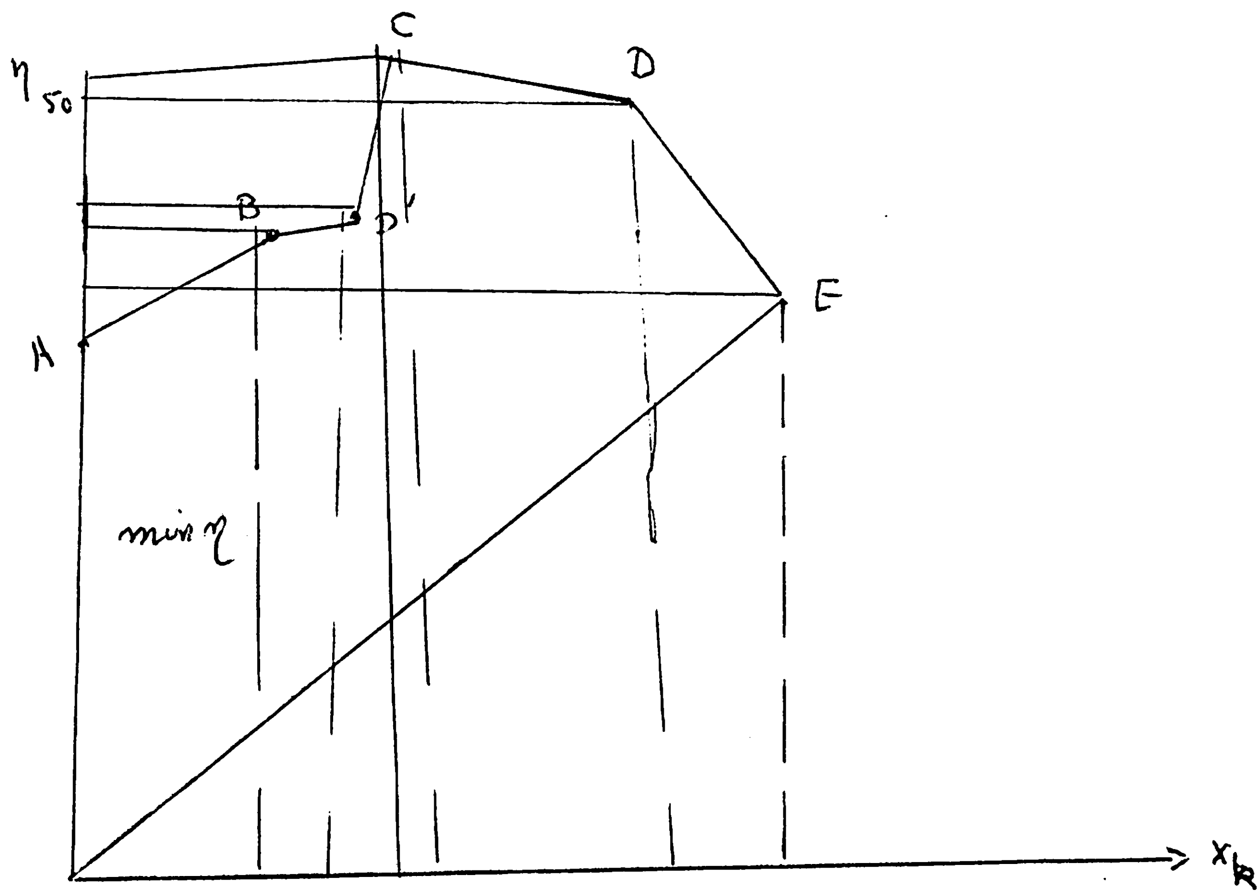


Fig (4)

Starting now from the solution point ($\eta = \eta^o$, $x_k = x_k^o$) of problem (36), (37), (38) pointc and moving through the upper boundes of the convex branches of the η - x_k plane in the direction of min x_k and max x_k untill we get integer value for x_k, this shows that the value of η^o is moved to smaller value.

Both integer values of x_k are (x_k^o) and $(x_k^o)+1$ to which the value of η are η_{rm} and η_{rm} (points D and D').

Now we take the greatest value of η with the coresponding value of x_k^o, then there is no better solution for the problem with an integer x_r.

Land and Doig used this systematic relation and derive the following method.

Now it is mecessary to define the following in solving many linear programming problem P (J'), means for our linear programming problem (36), (37), (38) in which the J' variables of integer conditions under 'J'=1.2, ... n'). S_j represents all the Set of feasible solutions and $\bar{S}_j$ are the Set of all non-feasible solutions of the problem P (J). In the special case where x_r and x_k integers, then we write P (2,r,k) and the set of feasible solutions as S_2 (r,k).

If S_n, is the solution of problem P (j') then the maximum value of is within the doman of the optimal solution, The value of such solution is always smaller than the value for

$S_{n'-i}$, $S_{n'-2'}$, ...

The greatest limit is S_o with η^o for problem P(o), in which no variables are integers.

The method is considered in the following steps.

i- The inilial point lay in the optimum of P(o)

This point is written as η^o and it is considered as intial point of a tree. If η^{o} is the solution of the problem P (n') then it is the required solution.

ii- If η^{o} is not the solution of P (n'), we begin to deal with problem P (i,r) in which the basic variable x_r is a solution coresponding to the value η^{o} and is non-integer we define.

(x_r^{o}) where (x_r^{o})= 1.

Now both linear programming problems are

P (0) with $x_r = (x_r^{o})$

P (0) with $x_r = (x_r^{o}) + 1$

are to be solved.

If there exsist a solution for both problems say rm rm then thess solution are elements of S,.

If both solutions are not Feasible then there is no solution of P (n') and we stop our calculations

iii- Now let $\eta' = \max(\eta_{rm}, \eta_{rm})$. The value of η' is the greatest value of the integerabilty of x_r. If x_r=V for η' then V must be equal to (x_r) or equal to (x_r) +1. This x_r = V form together with the problem P (o) a convex area in the $\eta - x_r$ plane. The next greatest value of η is given through the maximum values of x_r = V-1, x_r = V+1 also we have

$$\max \left[(x_r = V-1 . , \eta(x_r = V+1) \right]$$

the second best sloution or P (1;r).

(iv)- We deal now with the problem P (2 ,r,k). The variable x_k is basic variable for the solution . Again we deal with (x_k) and $(x_k) + 1$. IF both vlues for x_k are basic, then we bild η_{km}, η_{kM}. With maximum value η^2., If $\eta^2 > \max\left[\eta(x_r^r = V-1), \eta(x_r = V+1)\right]$ If η^2 is small as the second solution for $x_r = 0$ then our calculation is continuied.

(v) From those given values of the non-integer optimum the greatest value of or in case of integer variables can be optcined which is the optimal value for as a solution of P(n').

3-z Exampel:

We condider now the same problem solved by Gonery which is

$$\eta = 5x_1 + 6\,x_2 \;\ldots\ldots\ldots\ldots\ldots\ldots\; \max \qquad (39)$$

with

$$\begin{aligned} 4x_1 + x_2 &\leq 32 \\ 3x_1 + zx_2 &\leq 26 \\ 6x_1 + 8x_2 &\leq 67 \\ 6x_1 + 12x_2 &\leq 89 \end{aligned} \qquad (40)$$

and the non-negative constraints.

The non-infeger solution of this problem is

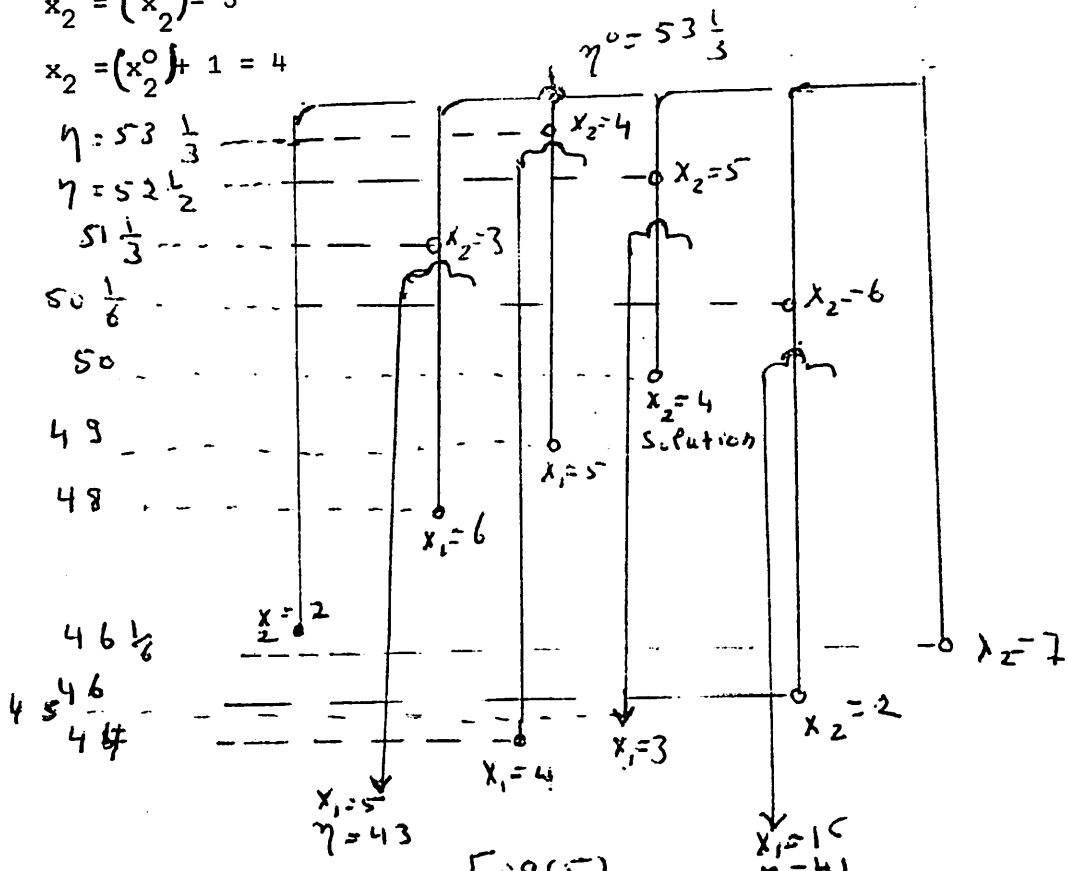
$$\begin{aligned} &x_1 = 6\tfrac{1}{6} \quad \bar{x}_1 = 3\tfrac{7}{12} \qquad \bar{x}_2 = \bar{x}_3 = 0 \\ &x_2 = 3\tfrac{3}{4} \quad \bar{x}_4 = 7 \qquad \eta = 53\tfrac{1}{3} \end{aligned} \qquad (41)$$

Applying now land and Doig method, so we get the following steps.

i) We deal now with the problem P(1;2) this means that the basic variable x_2 (in the solution η^o) must be integer this gives,

$x_2 = (x_2^o) = 3$

$x_2 = (x_2^o) + 1 = 4$



Fig(5)

The linear programm

(b) (39), (40), (41) with $x_2 = 3$ and (bb) (39), (40), (41) with $x_2 = 4$ are to be solved. The programms (b) and (bb) are formed in which the product $3a_{12}$, 4912 are subtracted from a_{io}. We get

$\eta_{2m} = 18 + 5 x_1$ .. .. max

$4x_1 \leq 32 - 3.1 = 29$

$3x_1 \leq 26 - 3.2 = 20$

$6x_1 \leq 67 - 3.8 = 43$ (b)

$6x_1 \leq 89 - 3.12 = 33$

$$x_1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \geqslant 0$$

$$\eta_{2M} = 24 + 5\,x_1 \qquad \max$$

$$4x_1 \leqslant 32 - 4.1 = 28$$

$$3x_1 \leqslant 26 - 4.2 = 18$$

$$6x_1 \leqslant 67 - 4.8 = 35 \qquad \text{(bb)}$$

$$6x_1 \leqslant 89 - 4.12 = 41$$

$$x_1, \bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4 \geqslant 0$$

For both programms (b), (bb) we get the following solutions

for (b) $x_1 = 6\frac{2}{3}$, $\bar{x}_, = 2\frac{1}{3}$ $\bar{x}_3 = 3$

$x_2 = 3$, $\bar{x}_2 = 0$ $\bar{x}_4 = 13$, $\eta_{2m} = 51\frac{1}{3}$

for (bb) $x_1 = 5\ 5/6$ $\bar{x}_1 = 4\ 2/3$ $\bar{x}_3 = 0$

$x_2 = 4$ $\bar{x}_2 = \frac{1}{2}$ $\bar{x}_4 = 6$ $\eta_{2M} = 53\frac{1}{6}$

Both value of x_2 are Feasible, they must be considered in the following solutions

ii) Now taking $\eta^1 = \max(\eta_{2m}, \eta_{2M})$

$$= \max 51\frac{1}{3}\ 1/3,\ 53\ \frac{1}{66}$$

i.e $\eta' = 53\frac{1}{6}$

and the best solution for x_2 are get after that

$$x_2 = 4 \min \eta'$$

In order to define this second solution (integer solution of x_2), we need only to test the value after $x_2 = 4$, the value of $x_2 = 3$ in (b) is good defined now we test for $x_2 = 5$ which gives the linear programm

$\eta = 30 + S x_1 \ldots \ldots \ldots \max$

$4x_1 \leq 32 - 5.1 = 27$

$3x_1 \leq 26 - 5.2 = 16$ (c)

$6x_1 \leq 67 - 5.8 = 27$

$6x_1 \leq 89 - 5.12 = 29$

as a solution we get

$x_1 = 4\frac{1}{2}, \bar{x}_1 = 9, x_3 = 0$ $\eta = 52\frac{1}{2}$

$x_2 = 5, \bar{x}_2 = 2\frac{1}{2}$ $\bar{x}_4 = 2$

the second best solution is

$$\eta = \max\left[\eta(x_2 = 3), \eta(x_2 = 5)\right]$$

with

$$\eta = \eta(x_2 = 5) = 52\frac{1}{2})$$

also now we get a set of values

$\eta^0 = 53\frac{1}{3}$ $\eta(x_2 = 5) = 52\frac{1}{2}$

$\eta^1(x_2 = 4) = 53\frac{1}{6}$ $\eta(x_2 = 3) = 51\frac{1}{3}$

iii) For the variable x_2 it is optimal programem with η' we go now to the linear programum P (2,2,1) this means that the values x_1, x_2 must take integer values.

Form the solution (bb) for η' we have $x_1 = 5\ 5/6$, also heir we put

$x_1 = (x_1^1) = 5^-$

$x_1 = (x_1^1) + 1 = 6$

Now we have to solve the programms

d) (39), (40), (41), with $x_2 = 4$, $x_1 = 5^-$

and

(dd) (39), (40), (41) with $x_2 = 4$, $x_1 = 6$

Since x_1 is the last integer variable, So we concentrat on testing the feasiblity of both values.

$x_1 = (x^1)$, $x_2 = 4$

$x_1 = (x_1^1) + 1$, $x_2 = 4$

the programm (dd) is not feasible, then we have only η^2 as a solution

$\eta^2 = 5,5 + 4.6 = 49$

with

$x_1 = 5^-$, $\bar{x}_1 = 8$, $\bar{x}_3 = S^-$

$x_2 = 4$, $\bar{x}_2 = 3$, $\bar{x}_4 = 9$

η^2 is a feasible integer solution for all the variables and also for values $x_2 = 4$ and $x_1 = 4$ we get $\eta = 44$

Now we deal with those given values of x_1 and x_2 from the solutions

$\eta^o = 53\frac{1}{3}$, $\eta(x_2 = S^-) = S^- 2\frac{1}{2}$

$\eta^1 (x_2 = 4) = 53\frac{1}{6}$, $\eta (x_2 = 3) = S1\frac{1}{3}$

$\eta^2 (x_2 = 4, x_1 = 5) = 49$

Since η^2 is smaller as $\eta(x_2 = 5)$ and $\eta(x_2 = 3)$ then η^2 not yet an optimal Solution,

iv) Now before taking the value $x_2 = S^-$ we must define the value of η for the values of $x_2 = 4$, $x_2 = 6$. For $x_2 = 4$ we have the corresponding value of η, but for $x_2 = 6$ we have the solution as.

$$x_1 = 2\ 5/6 \qquad \bar{x}_1 = 12\tfrac{2}{3} \qquad \bar{x}_3 = 0$$
$$x_2 = 6 \qquad \bar{x}_2 = S^-\tfrac{1}{2} \qquad \bar{x}_4 = 2$$
$$= S^-0\ 1/6$$

In general we still have three values for $(x_2 = S^-)$, $(x_2 = 3)$, $(x_2 = 6)$.
which are greater than η^2. Since $(x_2=5^-)$ is the greatest value of then we follow this way. From the programm (C) gives for $x_2 = S^-$, $x_1 = 4\frac{1}{2}$ i.e we have to test for

$x_2 = S^-$, $x_1 = 4$
$x_2 = S^-$, $x_1 = S^-$

The value $x_1 = S^-$, $x_2 = S^-$ is not feasible solution. For the values $x_1 = 4$, $x_2 = S^-$ we get the solution

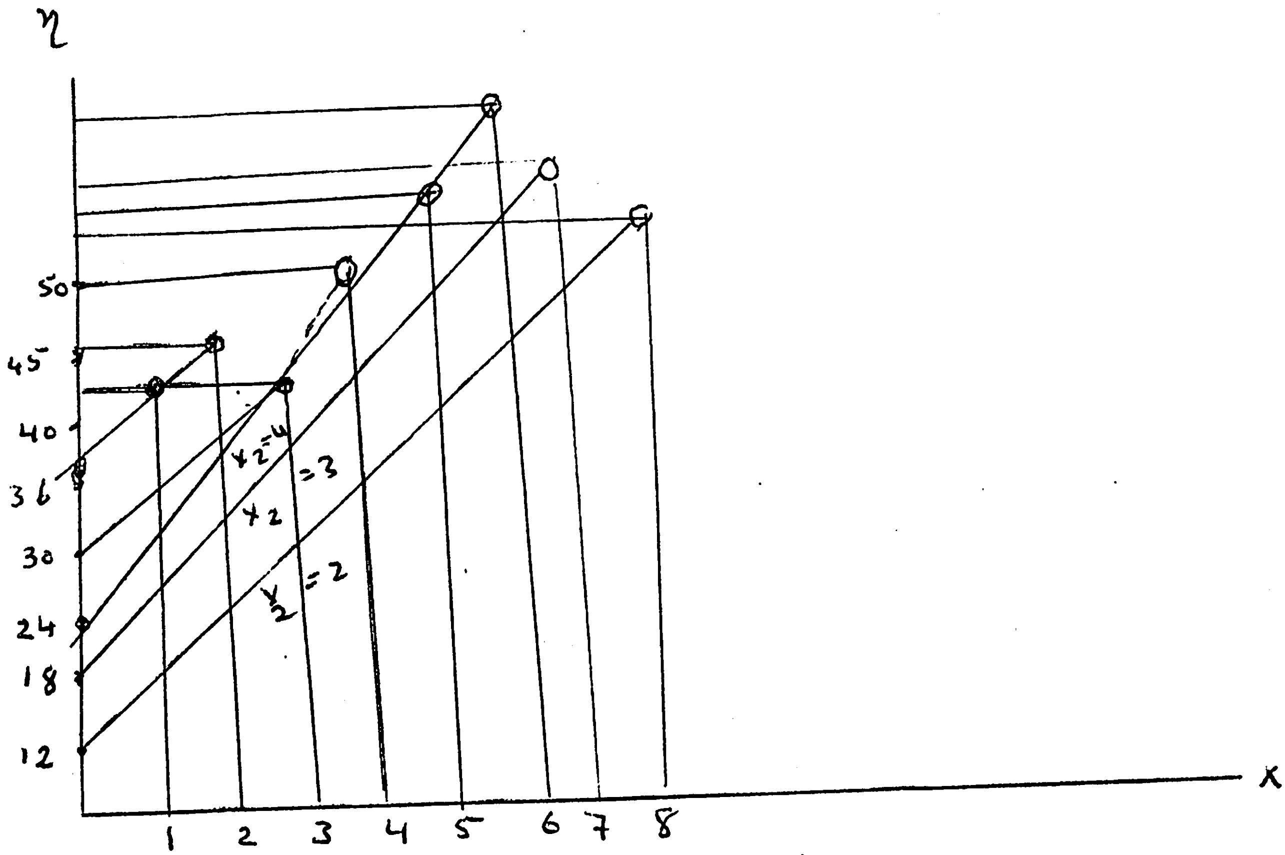
$$x_1 = 4 \qquad \bar{x}_1 = 11 \qquad \bar{x}_3 = 3$$
$$x_2 = S^- \qquad \bar{x}_2 = 4 \qquad \bar{x}_4 = S^- \qquad \eta = S^-0$$

Now we consider the following η values

$\eta\ (x_2 = S^-, x_1 = 4) = S^-0$
$\eta\ (x_2 = 3) \qquad = S^-1\ \frac{1}{3}$
$\eta\ (x_2 = 6) \qquad = S^-0\ 1/6$

To be tested is the value of $x_2 = 3$ and neben value $x_2 = 2$ The value of $\eta(x_2 = 2)$, $\eta\ (x = 3)$, $x_1 = 6)$, $\eta\ (x_2 = 3$, $x_1 = S^-)$ are smaller than that of $(x_2=S^-, x_1=4)$. The value of $(x_2=3, x_1=7)$ is not Feasbile. Fig (6) gives value to be tested. From which we get now the optimal solution of the problem with integer values of $x_2=S^-$ $x_1=4$ and $=S^-0$

Fig (6) shows the different solutions in the $-x_1$ plan and also the optimal solution.



Fig(6)

4- Comparasion of Gomory, Dantzig and land and Doig Methods:

4-1 Land and Daig method as a systematic method:

Form the previons discussion of the problem we see that land and Doig methed is a systematic method to obtaim an optimal integer solution. The meltod start by taking an integer value for one of the variables to get on optimal solution, then we begin dealing with the second variable and see the combination between the two variables in the integer cose to reach an optimal solution for the programm. untill the optimal integer solution is reached.. The solution is reached after a number of steps. which is not the cose by Dantzig method.

4-2 Combarsion of Gomary and Dantzing Methods:

Dantzig agree that Gomory constraints are more effeetive than those developed by himself the prove for that is as follows Gomary resterations are as

$$f_o \lesssim F_j \, x_j^N \qquad (42)$$

with $0 < f < 1$

(42) is get from the selation

$$x_i^B = 0$$

$$x_i^B = a_{io} - \sum_{j=1}^{n} a_{ij} x_j^N = 0 \qquad (43)$$

$$(i=0,\ 1,\ \ldots\ \ldots,\ m)$$

$$(x_o^B = 0)$$

If we deal only with equations then (42) is the simpel form of (43). Now the constraint of the form (42) is not obtained not only from (43) but also from the integer values of (43).

Let t be an integer number, then through multiplication (43) with t, so we get

$$tx^B = t_{ao} - \sum_{j=1}^{n} t_{aj} \, x_j \;\; -0 \qquad (44)$$

Separating the integer values $\bar{\eta}_j$ of t_{aj} and the restparts as F_j then we have

$$t_{aj} = \bar{n}_j + \bar{f}_j \qquad (0 \leq \bar{f}_j \leq 1)$$

and we get new restriation in the form

$$\bar{f}_o \leq \sum_{j=1}^{n} \bar{f}_j \, x_j^N \, . \qquad (4S^-)$$

If D is the smallest part of a_j $(i=0,1, \dots, n)$ then the restpart F_j for t=k and t=D+k are equal. In the special cose that t=D-1 we have

$$\bar{F}_j = \begin{cases} 1 - F_j & \text{for } 0 \leq \bar{F}_j < 1 \\ 0 & \text{for } F_j = 0 \end{cases}$$

taking $F_j > 0$ then $(4S^-)$ will be $1-F_o \leq \sum_{j=1}^{n} (1-F_j) \, x_j^N$. (46)

If we add to (46) the restraition (42)

$$F_o \leq \sum F_j \, x_j^N$$

then we get

$$1 \leq \sum x_j^N \qquad (47)$$

Relation (47) represents the condition for abtaiming an optimal integer solution due to Dantzig which are obtained by Gomory the through relations of the form (42) and (46) i-e relation(47) are included in Gomory method. Dantzig restraction is obtaimed with different ways not only that through the multiplication of (42) with)D-1). The non-integer rest part F_j (J= 0,1, ...n) of restraction (42) can be obtaimed through multiplication with -1, D-1, 2D-1, ... i-e

$$\bar{\bar{f}}_j = \begin{cases} 1-\bar{f}_j & \text{for } \bar{f}_j \quad 0 \\ 0 & \text{for } \bar{f}_j = 0 \end{cases} \tag{48}$$

Relations (48) is fullfiled with multiplication with values, which are more the 1 from D, also we have

$$(D+K)\ f_o \leqslant \sum_{j=1}^{n} f_j\ x_j^N\ (D+K),\ (k < D) \tag{49}$$

$$(D-K)\ f_o \leqslant \sum_{j=1}^{n} f_j\ x_j^N\ (D-K) \tag{50}$$

which gives also restractions of the form of Dantzig restraltions.

from (49) i-e (S^-0) we get for all $f_j > 0$, (j=0,1, ... n)

$$\bar{f}_o \leqslant \bar{f}_1\ x_1^N + \bar{f}_2\ x_2^N + \ldots \ldots \ldots \bar{f}_n\ x_n^N \tag{S^-1}$$

and

$$\bar{f}_o \leqslant \bar{\bar{f}}_1\ x_1^N + \bar{\bar{f}}_2\ x_2^N + \ldots \ldots \bar{f}_n\ x_n^N \tag{S^-2}$$

where

$\bar{f}_j$ and $\bar{\bar{f}}_j$, $\bar{n}_j$, $\bar{\bar{n}}_j$ are obtained through seperation of rest parts and integer number of (49).

Solving for (x_r^N) (S^-1) and (52) we get.

$$x_r^N \geqslant \frac{\bar{f}_o}{\bar{f}_r} - \frac{1}{\bar{f}_r}\ \bar{f}_j\ x_j^N \tag{S^-1'}$$

(j=1, ..., r-1, r+1, ... n)

$$x_r^N \geqslant \frac{1-\bar{\bar{f}}_o}{1-f_r} - \frac{1}{1-f_r} \sum (1-\bar{\bar{f}}_j)\ x_j^N \tag{S^-2'}$$

(j=1, ..., r-1, r+1, ... n)

Dantzig derive x_r^N from the restration (47) as

$$x_r^N \geq 1 - \sum_j x_j^N \qquad (47')$$

The value of $\bar{F}_o$, $\bar{F}_r$, $(1-\bar{F}_o)$; $(1-\bar{F}_r)$ and 1 gives the axis-Sections of the x_r^N axises- of the restrications $(5^-1')$, (5^-2) and (47).

Dontzig restractions bring the boundes polyder to a smaller area around the wnit . This smaller area is obtaimed from the . inequality of gomory at the x_r^N and $\bar{F}_o$: $\bar{F}_r$ i.e $(1-\bar{F}_o)$: $(1-\bar{F}_r)$ then

$$\bar{F}_o > \bar{F}_r \qquad (5^-3)$$

Which means that the boundes from (5-1) is more effective than that of Dantzig.

From (5^-3) we get

$$1 - \bar{F}_o < 1 - \bar{F}_r \qquad (5^-4)$$

Which says that the restrication $(5^-1)$is more effective than that of Dantzig, where from (5^-4) we have

$$\frac{1 - \bar{F}_o}{1 - \bar{F}_r} < 1$$

Which fubfill the oppsite relation

$$\bar{F}_o < \bar{F}_r \text{ , } (1-\bar{F}_o) \geq (1-\bar{F}_r) \qquad (5^-5^-)$$

then are the restrication $(5^-2')$ effectuve and $(5^-1')$ not effective as (47')

In case of $\bar{F}_o = \bar{F}_r$ and also $(1-\bar{F}_o) = (1-\bar{F}_r)$, rhen the three restrication of the x_r^N - axis have the Same value Fig . (7) give the relation between $\bar{F}_o$; $\bar{F}_r$ and $(1-\bar{F}_o):(1-\bar{F}_r)$

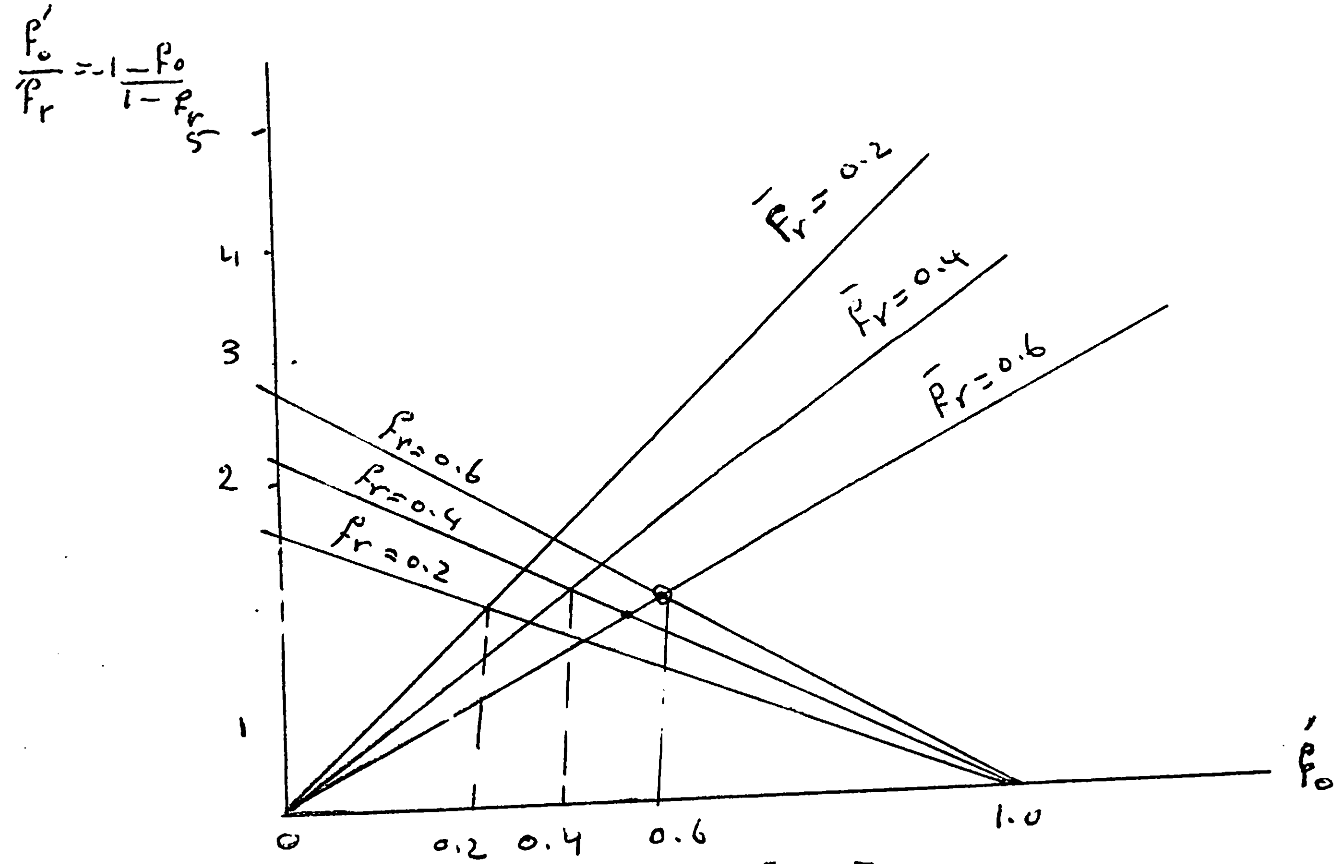


Fig (7)

If is clear that for all values of $\bar{F}_o$ and $\bar{F}_r$ the functions $\bar{F}_o;\bar{F}_r$ and $(1-\bar{F}_o)$: $(1-\bar{F}_r)$ always intersets at the point $1=\bar{F}_o : \bar{F}_r= (1-\bar{F}_o) : (1-\bar{F}_r)$ where $\bar{F}_o > \bar{F}_r$ for the intersected point

$$\frac{\bar{F}_o}{\bar{F}_r} < 1, \qquad \frac{1-\bar{F}_o}{1-\bar{F}_r} > 1$$

also

$$\frac{\bar{F}_o}{\bar{F}_r} > 1, \qquad \frac{1-\bar{F}_o}{1-\bar{F}_r} < 1$$

Also Fig (7) shaws that Gomory method is more convergence than that of Dantzig method.

For the relations (49) and (5‾0) we can write also

$$(D+k_1)\ f_0 \lesssim \sum_j f_j x_j^N\ (D+k_1) \qquad (5^-6)$$

$$(D+k_2)\ f_o \lesssim \sum f_j x_j^N\ (D+k_2)$$

and

$$(D-1_1)\ f_o \lesssim \sum f_j x_j^N\ (D+1_1) \qquad (5^-7)$$

$$(D-1_3)\ f_o \lesssim \sum f_j x_j^N\ (D-1_2)$$

where

$$\sum k_i = \sum 1_i$$

4-3 An Exampel for companing the metheds

In the exampel of Gomory method and that of Dantzig method we have for the non-basic restrication of the non-integer optimal toble that

$$\frac{7}{12} \leqslant \frac{5^-}{12}\,\bar{x}_3 + \frac{10}{12}\,\bar{x}_2 \qquad (5^-8)$$

Through multiplication of this restraction by the integer numbers 2, 3, ... , 11 then

we get

$$\frac{11}{12} \leqslant \frac{1}{12}\,\bar{x}_3 + \frac{2}{12}\,\bar{x}_3 \qquad (5^-9)$$

Form (5¯9) and with the slack variable S_1 we can get a feasible solution for primal and dual, heir must be other restriaetion in the form

$$\frac{5^-}{6} \leqslant \frac{1}{6}\,\bar{x}_4$$

with the slack variable S_2

Now we are in a sitution that the primal and daul have a feasible solution.

Relution (5¯9) and(60) can be represented by the following Fig (8)

For $\bar{x}_4$ in (60) we have

$$\bar{x}_4 = 7 + 2\,\bar{x}_3 - 2\,\bar{x}_2$$

Recation (60) give then

$$\frac{5^-}{6} \leq \frac{7}{6} + \frac{2}{6}\bar{x}_3 - \frac{2}{6}\bar{x}_2$$

$$\frac{1}{6} \geq - \frac{2}{6}\bar{x}_3 + \frac{2}{6}\bar{x}_2$$

Fig (8) shows that the feasible area is (due to the two new restrica-tion) smaller than that of Fig (1) and (2).

The Frist inequality in Gomory method is

$$\frac{7}{12} \leq \frac{5^-}{12}\bar{x}_3 + \frac{10}{12}\bar{x}_2 \qquad (61)$$

If we multiply (61) with D - 2 = 10 and D + 2 = 14 then we get

$$(D-2)\,.\ \frac{10}{12} \leq \frac{2}{12}\,\bar{x}_3 + \frac{4}{12}\,\bar{x}_2$$

$$(D+2)\,.\ \frac{2}{12} \leq \frac{10}{12}\,\bar{x}_3 + \frac{8}{12}\,\bar{x}_2\ .$$

The sum will be equal to one which isDantzig restriaction.

Fig (8) shows that the optimal solution is at the point H.

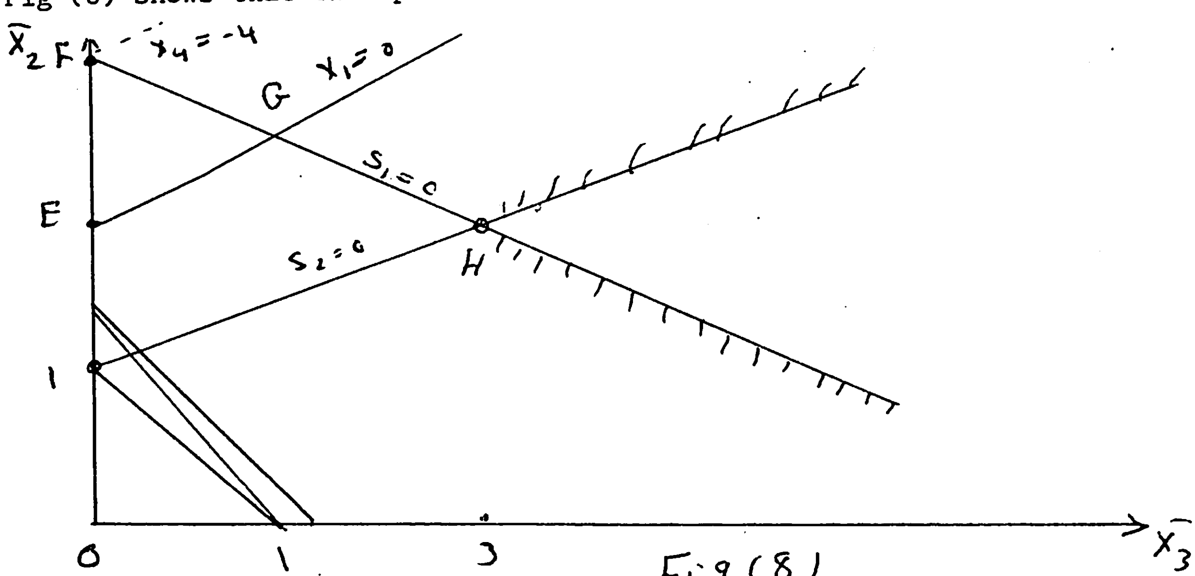


Fig (8)