د. سيد ابو الليل

ARAB REPUBLIC OF EGYPT

THE INSTITUTE OF NATIONAL PLANNING

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Update the Transportation Problem

by

Dr. Afaf Fouad Nakhla
and
Rafat Riad Riskalla

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Introduction:

The transportation problem is of particular interest because of its economic application and its computational simplicity. The transportation problem is commonly used to describe a particular type of linear programming problem, which has the following special structure:

$$\text{Minimize} \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} C_{ij} \; X_{ij}$$

$$\text{Subject to} \quad \sum_{j=1}^{n} X_{ij} = a_i, \quad i=1, 2, \ldots, m \qquad \text{(I)}$$

$$\sum_{i=1}^{m} X_{ij} = b_j, \quad i = 1, 2, \ldots, n$$

$x_{ij} \geq 0$, $i=1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

The above linear programming problem may be considered as one in which various amounts of a commodity are to be shipped from each of m "origins" to each of n "destinations", The amount available for shipment from the i th origin is a_i, i= 1, 2, ..., m; the amount required by the j th destination is b_j, j = 1, 2, ..., n. The cost of shipping each unit from origin i to destination j is c_{ij}. Obviously, X_{ij} denotes the quantity to be shipped from origin i to destination j. In order for this problem to have a feasible solution, the total supply must equal the total demand. i.e.

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{m} b_j .$$

The dual of the problem (I)* can be written as:

$$\underline{\text{Maximize}} \quad W = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j V_j$$

Subject to

$$U_i + V_j \leqslant c_{ij}$$

$$i=1,2,\ldots,m \text{ and } j=1,2,\ldots,n.$$

The UV-method is one of several methods used to solve the transportation problem.

Now we define the updated transportation problem:

Suppose that some of the quantities $U_i + V_j$, $i=1, 2, \ldots, m$; $j=1, 2, \ldots, n$ have negative integer values. We show in the sequal that with the same cost matrix, We can ship more quantities from certain origins to certain destinations, which are associated with those cells corresponding to the negative quantities $U_i + V_j$, with less cost than the cost of shipping the quantities a_i and b_j of the original problem (I). The new problem is called the updated transportation problem having the increased amount of a_i and b_j with less optimal value for the objective function.

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* See [1] and [2].

In this paper we present a new algorithm for solving the updated transportation problem. The algorithm is based on the UV-method. Section one deals with some special properties for the updated problem. The algorithm is presented in section 2. In section 3 we give an illustrative example pertaining to the algorithm. We modify the computer program in [3] of the standard transportation problem to adapt our updated transportation problem*. The program is used to run many test problems of different sizes. In the appendix, we present the results of some of these test problems.

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* See [7].

1. Some properties:

1. If Z_1 and Z_2 are the total costs of the optimal solutions to the original problem and its updated problem respectively, then $Z_2 < Z_1$.

Proof:

For the original problem, let (s,t) be the empty cell in which,

$$R_{st} = \min R_{ij} < 0$$

and

$$\Delta \max > 0.$$

Then,

$$Z_1 = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

and,

$$Z_2 = \sum_{i=1}^{m} a_i' u_i + \sum_{j=1}^{n} b_j' v_j$$

where,

$$a_i' = a_i \quad \text{for all } i \neq s$$

and,

$$a_s' = a_s + \Delta \max$$

$$b_j' = b_j \quad \text{for all } j \neq t,$$

$$b_t' = b_t + \Delta \max$$

Therefore,

$$Z_2 = \sum_{\substack{i \neq s \\ i=1}}^{n} a_i U_i + (a_s + \Delta \max) U_s + \sum_{\substack{j \neq t \\ j=1}}^{n} b_j V_j + (b_t + \Delta \max) V_t$$

$$Z_2 = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j + (u_s + V_t) . \Delta max$$

$$= Z_1 + R_{st} . \Delta max$$

$$\therefore \quad Z_2 < Z_1$$

Since $R_{st} < 0$ and $\Delta max > 0$.

2. If all a_i and b_j are integers, then Δmax is also an integer.

Let P(1) denote the original problem and Z[P(1)] be the total cost of its solution.

Starting from the problem P(k) in which $R_{st} = \min R_{ij} < 0$ and $\Delta max > 0$. There is a sequence of updated problems $\{P(K+r)\}_{r=1}^{\Delta max}$ satisfying the following:

i. $Z[P(k+r)] < Z[P(k+r-1)]$ for all $0 < r \leq \Delta max$

and,

$Z[P(k+r)] = Z[P(k+r-1)] + r.R_{st}$ for all $0 < r \leq \Delta max$

ii. $P(k+\Delta max)$ is the best updated problem to P(k) in this sequence

iii. The optimal dual solutions to problems P(k) and P(k+r) are the same for every $0 < r \leq \Delta max$.

2. Description of the algorithm:

Set K = 0

Step 1: Start from the optimal primal and dual solutions to the updated problem number k in which the total cost is Z.

calculate:

$R_{ij} = u_i + v_j$ for all the empty cells still under consideration.

Step 2: Choose the empty cell (s,t), according to:

$R_{st} = \min R_{ij}$

If $R_{st} < 0$, go to step 3, otherwise stop, because the problem cannot be updated more.

Step 3: Determine the values $y_{st}^{\alpha\beta}$ according to*:

$$y_{ij}^{\alpha\beta} = \begin{cases} +1 & , \text{ if t is even} \\ -1 & , \text{ if t is odd.} \end{cases}$$

or by solving the system of the basic variables for all $a_\alpha = 0$ except $a_s = 1$ and all $b_\beta = 0$ except $b_t = 1$.

Step 4: Choose the basic variable $x_{qr}^{\beta} = \Delta\max = \min \left\{ x_{\alpha\beta}^{B} : y_{st}^{\alpha\beta} = -1 \right\}$

If $\Delta\max = \varepsilon$. remove the cell (s,t) from further consideration and return to step 1, otherwise set k=k+1 and go to step 5.

Step 5: (The updated problem number k)

Update the problem as follows:

(*) See [5] and [7].

$$a_i' = a_i \quad \text{for all } i \neq s,$$

$$a_s' = a_s + \Delta\max,$$

$$b_j' = b_j \quad \text{for all } j \neq t$$

$$b_t' = b_t + \Delta\max$$

The values of the new basic variables are computed by:

$$X'^{B}_{\alpha\beta} = \begin{cases} X^{B}_{\alpha\beta} - \Delta\max, & \text{if } y^{\alpha\beta}_{st} = -1 \\ X^{B}_{\alpha\beta}, & \text{if } y^{\alpha\beta}_{st} = 0 \\ X^{B}_{\alpha\beta} + \Delta\max, & \text{if } y^{\alpha\beta}_{st} = +1 \end{cases}$$

$X'^{B}_{\alpha\beta} = \varepsilon$ for the basic variables with zero value

(i.e. degenerate basic variables)

Now the total cost is

$$Z = Z + R_{st}\,\Delta\max.$$

Remove the cell (s,t) from further consideration and go to step 1.

3. Illustrative example:

Consider a transportation problem with three warehouses and four markets. The warehouses capacities are $a_1 = 10$, $a_2=15$ and $a_3 = 20$. The market demands are $b_1=5$, $b_2 = 12$, $b_3=13$ and $b_4=15$. The unit cost of shipping is given by the following table.

	M_1	M_2	M_3	M_4
W_1	25	10	2	30
W_2	5	15	20	10
W_3	100	65	50	2

The transportation table is given by:

Markets / warehouses	M_1	M_2	M_3	M_4	Supplies a_i
W_1	25 x_{11}	10 x_{12}	2 x_{13}	30 x_{14}	10
W_2	5 x_{21}	15 x_{22}	20 x_{23}	10 x_{24}	15
W_4	100 x_{31}	65 x_{32}	50 x_{33}	2 x_{34}	20
Demands b_j	5	12	13	15	

The optimal solution to this problem is given by the shipments shown in fig (1).

U_i \ V_j	0	10	2	-46	a_i
0	25	10, 2	2, 8	30	10
5	5, 5	15, 10	20	10	15
48	100	65	50, 5	2, 15	20
b_j	5	12	13	15	

Fig. 1

The cost of this solution is 491 and the total amount transported is 45 units.

Now, consider the problem with the same cost matrix but with supply and demands given by the vectors.

$$a = (\ 11,\ 15,\ 20\ )$$

$$b = (\ 5,\ 12,\ 13,\ 16\ )$$

The optimal solution is given by the shipments shown in fig. 2)

U_i \ V_j	0	10	2	-46	a_i
0	25	10, 2	2, 9	30	11
5	5, 5	15, 10	20	10	15
48	100	65	50, 4	2, 16	20
b_j	5	12	13	16	

Fig. (2)

The cost of this solution is 445 and the total amount transported is 46 units.

We notice that the total cost has been reduced, although we transport more than before.

From the optimal solution in fig (1), we have $U_1 + V_4 = -46 < 0$. Then an increase of a_1 and b_4 by the same amount, will decrease the total cost, what we want to investigate is how far can this go on?.

Let us return to the optimal solution of the first problem and assume that a_1 and a_4 are increased by the same amount Δ. The circuit containing the empty cell (1,4) is determined as shown in fig (3):

U_i \ V_j	0	10	2	-46	a_i
0	(25)	(10) 2	(2) 8	(30) *	10
5	(5) 5	(15) 10	(20)	(10)	15
48	(100)	(65)	(50) 5	(2) 15	20
b_j	5	12	13	15	

Fig. (3)

add and subtract the amount Δ which is the basic variable, on this circuit such that the rows and columns constraints are satisfied. The largest permissible value of Δ is that which first makes one of the basic variables zero. Then,

$$\Delta = x_{33} = 5$$

In this case, we can increase the value of a_1 and b_4 by the same amount $\Delta = 5$. The optimal solution is given by the shipments shown in fig. (4).

U_i \ V_j	0	10	2	-46	a_i
0	25	10 2	2 13	30	15
5	5	15 10	20	10	15
48	100	65	50 0	2 20	20
b_j	5	12	13	20	

Fig. (4)

The cost of this solution is 261 and the total amount transported is 50 units.

A similar result would be obtained if the increase were spread over a number of a_i's and b_j's, when the corresponding (u_i+V_j)'s are negative.

In fig(4), we notice that $(U_2 + V_4) = -41<0$, but we cannot form a circuit containing the corresponding empty cell (2,4). Therefore we cannot transport nomore.

5.3 The Computer Program*

```
C
C      STANDARD TRANSPORTATION PROBLEM
C
       DIMENSION C(15,15),X(15,15),A(15),B(15),U(15),V(15)
      *,NBI(15),NBJ(15),IJSSD(15,15)
       EPS=1.0E-10
C      CALL OPEN(9,"RRA",2,IER)
     1 READ(9,5)KASE,M,N,NATURE
     5 FORMAT(4I4)
C      M=NO.OF SOURCES(ROWS)
C      N=NO.OF DESTINATIONS(COLUMNS)
C      NATURE=0 FOR STANDARD TRANSPORTATION PROBLEM
C      OTHERWISE FOR CASE OF SLACKS IN ROW-AND COL.-EQS.
C      EPS=VERY SMALL QUANTITY APPROX.=ZERO.
       WRITE(12,6)KASE
     6 FORMAT(1H1,5X,'CASE NO.',I3/1H ,5X ,11(1H=))
       READ(9,10)(A(I),I=1,M)
       READ(9,10)(B(J),J=1,N)
    10 FORMAT(8F10.3)
       SUMA=0
       DO 15 I=1,M
    15 SUMA=SUMA+A(I)
       SUMB=0
       DO 20 J=1,N
    20 SUMB=SUMB+B(J)
       IF(NATURE.EQ.0)GO TO 30
C      CASE OF SLACKS IN ROW-AND COLUMN-EQUATIONS.
       WRITE(12,25)
    25 FORMAT(1H ,17X,'SLACKS IN ROW-AND COLUMN-EQUATIONS'/)
       M=M+1
       N=N+1
       A(M)=SUMB
       B(N)=SUMA
       SUMA=SUMA+A(M)
       SUMB=SUMB+B(N)
       GO TO 55
    30 IF(SUMA-SUMB)35,55,45
C      CASE OF UNDER-PRODUCTION.
    35 WRITE(12,40)
    40 FORMAT(1H ,17X,'UNDER-PRODUCTION'/)
       M=M+1
```

* The major port of this program ix taken from (3)

```
      A(M)=SUMB-SUMA
      SUMA=SUMA+A(M)
      GO TO 55
C     CASE OF OVER-PRODUCTION.
   45 WRITE(12,50)
   50 FORMAT(1H ,17X,'OVER-PRODUCTION'/)
      N=N+1
      B(N)=SUMA-SUMB
      SUMB=SUMB+B(N)
   55 DO 60 I=1,M
      READ(9,10)(C(I,J),J=1,N)
   60 WRITE(12,65)(C(I,J),J=1,N),A(I)
   65 FORMAT(1H ,12F8.2/)
      WRITE(12,65)(B(J),J=1,N)
      ITER=0
      CALL INITIA(C,M,N,A,B,EPS,X)
  100 WRITE(12,105)ITER
  105 FORMAT(1H ,5X,'ITERATION NO.',I3/1H ,5X,16(1H-))
      OF=0
      DO 120 I=1,M
      DO 120 J=1,N
      IF(X(I,J).EQ.0)GO TO 120
      WRITE(12,115)I,J,X(I,J)
  115 FORMAT(1H ,5X,'X(',I2,',',I2,') =',E12.6)
      OF=OF+C(I,J)*X(I,J)
  120 CONTINUE
      WRITE(12,125)OF
  125 FORMAT(1H ,28X,'O.FUNCTION =',E12.6,2X,'UNITS OF COST')
C     COMPUTATION OF THE SIMPLEX-MULTIPLIERS
      CALL UV(C,X,M,N,NBI,NBJ,U,V)
      CALL XSSD(C,X,M,N,U,V,CDIJ,IS,JSD)
      TYPE"CDIJ=",CDIJ
C     ARE ALL Z(I,J)-C(I,J).LE.ZERO.
C     IF YES,WE HAVE AN OPTIMAL SOLUTION.
C     IF NO,WE CAN REPEAT THE PROCESS BY BRINGING X(S,T)INTO THE BASIS
      IF(CDIJ.LE.0.)GO TO 150
      DO 140 I=1,M
  140 A(I)=0
      A(IS)=1.
      DO 145 J=1,N
  145 B(J)=0
      B(JSD)=1.
```

```
      CALL AIJSSD(X,M,N,A,B,NBI,NBJ,IJSSD)
      CALL CHANGE(X,IJSSD,M,N,IS,JSD,EPS)
C     IS,JSD ARE THE SUBSCRIPTS OF THE NEW ENTERING BASIC VARIABLE.
      ITER=ITER+1
      GO TO 100
C     OPTIMAL SOLUTION
  150 WRITE(12,155)SUMA,OF
  155 FORMAT(1H ,53X,'( OPTIMAL SOLUTION )'/1H ,15X,'IN THIS PROBLEM
     X WE TRANSPORT (',F10.3,') UNIT WITH TOTAL'/1H ,5X,'COST
     X(',E12.6,')')
      WRITE(12,160)
  160 FORMAT(1H1,21X,'UPDATE THE TRANSPORTATION PROBLEM'/1H ,21X,33(1H=))
C     UPDATE THE TRANSPORTATION PROBLEM
C     COMPUTE THE VALUES U(I)+V(J)
      DO 165 I=1,M
      DO 165 J=1,N
      IF(X(I,J).NE.0)GO TO 165
      C(I,J)=U(I)+V(J)
  165 CONTINUE
C     COMPUTE MIN. U(I)+V(J)
      IPDAT=0
  170 RRR=1.0E+38
      DO 175 I=1,M
      DO 175 J=1,N
      IF(X(I,J).NE.0)GO TO 175
      IF(RRR.LE.C(I,J))GO TO 175
      RRR=C(I,J)
      IR=I
      JC=J
  175 CONTINUE
      IF(RRR.LT.0)GO TO 190
      WRITE(12,180)
  180 FORMAT(1H ,15X,'THIS PROBLEM CAN NOT BE UPDATED MORE'/)
      GO TO 1
  190 DO 195 I=1,M
  195 A(I)=0
      A(IR)=1.
      DO 200 J=1,N
  200 B(J)=0
      B(JC)=1.
      CALL AIJSSD(X,M,N,A,B,NBI,NBJ,IJSSD)
```

```
C     COMPUTE DELTA=MIN.(BASICX(I,J))FOR IJSSD.LT.0.
      DELTA=1.0E+38
      DO 210 I=1,M
      DO 210 J=1,N
      IF(IJSSD(I,J).GE.0)GO TO 210
      IF(DELTA.LE.X(I,J))GO TO 210
      DELTA=X(I,J)
      IF(DELTA.EQ.EPS)GO TO 300
  210 CONTINUE
C     COMPUTE THE VALUES OF THE BASIC VARIABLES IN THE NEW
C     PROBLEM.
      DO 240 I=1,M
      DO 240 J=1,N
      IF(X(I,J).EQ.0)GO TO 240
      IF(IJSSD(I,J))230,240,220
  220 X(I,J)=X(I,J)+DELTA
      GO TO 240
  230 X(I,J)=X(I,J)-DELTA
      IF(X(I,J).NE.0)GO TO 240
      X(I,J)=EPS
  240 CONTINUE
      SUMA=SUMA+DELTA
      IPDAT=IPDAT+1
      WRITE(12,250)IPDAT
  250 FORMAT(1H ,5X,'THE UPDATED PROBLEM NO.(',I3,')'/1H ,5X,28(1H-))
      WRITE(12,260)IR,JC,DELTA
  260 FORMAT(1H ,15X,'WHEN WE INCREASE A(',I2,') AND B(',I2,') BY THE SAM
     XE AMOUNT'/,1H ,5X,'(',F8.2,') UNIT WE FIND A BETTER SOLUTION.')
      DO 270 I=1,M
      DO 270 J=1,N
      IF(X(I,J).EQ.0)GO TO 270
      WRITE(12,115)I,J,X(I,J)
  270 CONTINUE
      OF=OF+RPR*DELTA
      WRITE(12,125)OF
      WRITE(12,155)SUMA,OF
  300 C(IR,JC)=1.0E+38
      GO TO 170
      CALL EXIT
      END
```

```
C
C     INITIAL BASIC FEASIBLE SOLUTION
C     LEAST COST METHOD
C
      SUBROUTINE INITIA(C,M,N,A,B,EPS,X)
      DIMENSION C(15,15),X(15,15),A(15),B(15)
C
      NM1=0
C     NO.OF BASIC VARIABLES
      NROWSL=M
C     NO.OF ROWS LEFT
      NCOLSL=N
C     NO.OF COLUMNS LEFT
      DO 5 I=1,M
      DO 5 J=1,N
    5 X(I,J)=0
   10 IF(NM1.EQ.(M+N-1))GO TO 500
      CPQ=1.0E+38
      DO 20 I=1,M
      IF(A(I).LE.EPS)GO TO 20
      DO 20 J=1,N
      IF(B(J).LE.EPS)GO TO 20
      IF(CPQ.LE.C(I,J))GO TO 20
      CPQ=C(I,J)
      IP=I
      JQ=J
   20 CONTINUE
      IF(A(IP)-B(JQ))30,40,90
C     A(I).LT.B(J)
   30 X(IP,JQ)=A(IP)
      B(JQ)=B(JQ)-A(IP)
      A(IP)=0
      NM1=NM1+1
      NROWSL=NROWSL-1
      GO TO 10
C     A(I).EQ.B(J)
   40 IF(NROWSL-1)80,50,70
   50 IF(NCOLSL-1)80,60,65
C     ONLY ONE ROW AND ONE COLUMN LEFT
   80 X(IP,JQ)=A(IP)
      A(IP)=0
      B(JQ)=0
      GO TO 500
```

```
C      ONE ROW AND SEVERAL COLUMNS LEFT.
   65 X(IP,JQ)=B(JQ)
      B(JQ)=0
      A(IP)=EPS
      NM1=NM1+1
      NCOLSL=NCOLSL-1
      NROWSL=NROWSL-1
      GO TO 10
C      MORE ROWS.
   70 IF(NCOLSL.LT.1)STOP
C      MORE ROWS AND ONE OR MORE COLUMNS.
      X(IP,JQ)=A(IP)
      A(IP)=0
      B(JQ)=EPS
      NM1=NM1+1
      NROWSL=NROWSL-1
      NCOLSL=NCOLSL-1
      GO TO 10
   80 STOP
C      A(I).GT.B(J)
   90 X(IP,JQ)=B(JQ)
      A(IP)=A(IP)-B(JQ)
      B(JQ)=0
      NM1=NM1+1
      NCOLSL=NCOLSL-1
      GO TO 10
C      RESOLUTION OF DEGENERACY
  500 DO 510 I=1,M
      IF(A(I).EQ.0)GO TO 510
      IF(X(I,JQ).NE.0)GO TO 510
      X(I,JQ)=EPS
  510 CONTINUE
      DO 520 J=1,N
      IF(B(J).EQ.0)GO TO 520
      IF(X(IP,J).NE.0)GO TO 520
      X(IP,J)=EPS
  520 CONTINUE
      RETURN
      END
```

```
C
C     SIMPLEX MULTIPLIERS
C
      SUBROUTINE UV(C,X,M,N,NBI,NBJ,U,V)
      DIMENSION C(15,15),X(15,15),NBJ(15),U(15),V(15),NBI(15)
      NUV=0
C     NO.OF SIMPLEX MULTIPLIERS.
C     NO.OF BASIC VARIABLES IN EACH ROW AND EACH COLUMN.
      DO 3 J=1,N
    3 NBJ(J)=0
      DO 5 I=1,M
      NBI(I)=0
      DO 5 J=1,N
      IF(X(I,J).EQ.0)GO TO 5
      NBI(I)=NBI(I)+1
      NBJ(J)=NBJ(J)+1
    5 CONTINUE
C     ONE OF U,S OR ONE OF V,S MUST EQUAL ZERO
C     CHOOSE THAT ONE ASSOCHIATED WITH THE LARGEST NO. OF
C     BASIC VARIABLES.
      LNBI=0
C     THE ROW WHICH HAS THE LARGEST NO. OF BASIC VARIABLES.
      DO 15 I=1,M
      IF(LNBI.GE.NBI(I))GO TO 15
      LNBI=NBI(I)
      II=I
   15 CONTINUE
      LNBJ=0
C     THE COLUMN WHICH HAS THE LARGEST NO.OF BESIC VARIABLES.
      DO 25 J=1,N
      IF(LNBJ.GE.NBJ(J))GO TO 25
      LNBJ=NBJ(J)
      JJ=J
   25 CONTINUE
      IF(LNBI.GE.LNBJ) GO TO 200
C     FIRST STEP IN EVALUATING U IN CASE OF V=0
      V(JJ)=0
      NBJ(JJ)=0
C     MEANS THAT V(JJ) BECAME KNOWN.
      NUV=NUV+1
      DO 110 I=1,M
      IF(X(I,JJ).EQ.0)GO TO 110
```

```
      U(I)=C(I,JJ)
      NBI(I)=0
C     MEANS THAT U(I) BECAME KNOWN.
      NUV=NUV+1
  110 CONTINUE
      GO TO 215
C     FIRST STEP IN EVALUATING V IN CASE OF U=0
  200 U(II)=0
      NBI(II)=0
C     MEANS THAT U(II) BECAME KNOWN.
      NUV=NUV+1
      DO 210 J=1,N
      IF(X(II,J).EQ.0) GO TO 210
      V(J)=C(II,J)
      NBJ(J)=0
C     MEANS THAT V(J) BECAME KNOWN.
      NUV=NUV+1
  210 CONTINUE
  215 IF(NUV.GE.(M+N-1))GO TO 300
C     EVALUATION OF U KNOWING V.
      DO 240 I=1,M
      DO 240 J=1,N
      IF(NBI(I).EQ.0) GO TO 240
      IF(NBJ(J).NE.0) GO TO 240
      IF(X(I,J).EQ.0) GO TO 240
      U(I)=C(I,J)-V(J)
      NBI(I)=0
      NUV=NUV+1
  240 CONTINUE
      IF(NUV.GE.(M+N-1)) GO TO 300
C     EVALUATION OF V KNOWING U.
      DO 265 J=1,N
      DO 265 I=1,M
      IF(NBJ(J).EQ.0)GO TO 265
      IF(NBI(I).NE.0) GO TO 265
      IF(X(I,J).EQ.0)GO TO 265
      V(J)=C(I,J)-U(I)
      NBJ(J)=0
      NUV=NUV+1
  265 CONTINUE
      GO TO 215
  300 RETURN
      END
```

```
C     THE ENTERING BASIC VARIABLE
C
      SUBROUTINE XSSD(C,X,M,N,U,V,CDIJ,IS,JSD)
      DIMENSION C(15,15),X(15,15),U(15),V(15)
      CDIJ=-1.0E+38
C     MAXIMUM OF Z(I,J)-C(I,J)FOR X(I,J)=0
      DO 15 I=1,M
      DO 15 J=1,N
      IF(X(I,J).NE.0)GO TO 15
      CUV=U(I)+V(J)-C(I,J)
      IF(CDIJ.GE.CUV)GO TO 15
      CDIJ=CUV
      IS=I
      JSD=J
   15 CONTINUE
      RETURN
      END


C
C     THETA ADJUSTMENT.
C
      SUBROUTINE AIJSSD(X,M,N,A,B,NBI,NBJ,IJSSD)
      DIMENSION IJSSD(15,15),X(15,15),A(15),B(15),NBI(15),NBJ(15)
      NM1=0
      DO 10 J=1,N
   10 NBJ(J)=0
      DO 15 I=1,M
      NBI(I)=0
      DO 15 J=1,N
      IJSSD(I,J)=2
      IF(X(I,J).EQ.0)GO TO 15
      NBI(I)=NBI(I)+1
      NBJ(J)=NBJ(J)+1
   15 CONTINUE
C     WE SUPPOSE THAT IJSSD=2BECAUSE IT MUST BE=-1,0,
C     OR+1 ONLY AT THE END OF COMPUTATION.
   25 DO 50 I=1,M
      IF(NBI(I).NE.1)GO TO 90
```

```
C     AROW CONTAINS ONLY ONE BASIC VARIABLE.
      DO 40 J=1,N
      IF(X(I,J).LE.0)GO TO 40
      IF(IJSSD(I,J).GT.1)GO TO 45
   40 CONTINUE
   45 IJSSD(I,J)=A(I)
      B(J)=B(J)-A(I)
      A(I)=0
      NBI(I)=0
      NBJ(J)=NBJ(J)-1
      NM1=NM1+1
   50 CONTINUE
      IF(NM1.GE.(M+N-1))GO TO 100
      DO 80 J=1,N
      IF(NBJ(J).NE.1)GO TO 80
      DO 70 I=1,M
      IF(X(I,J).LE.0)GO TO 70
      IF(IJSSD(I,J).GT.1)GO TO 75
   70 CONTINUE
   75 IJSSD(I,J)=B(J)
      A(I)=A(I)-B(J)
      B(J)=0
      NBJ(J)=0
      NBI(I)=NBI(I)-1
      NM1=NM1+1
   80 CONTINUE
      IF(NM1.LT.(M+N-1))GO TO 25
  100 RETURN
      END
```

```
C
C     CHANGING THE BASIS.
C
      SUBROUTINE CHANGE(X,IJSSD,M,N,IS,JSD,EPS)
      DIMENSION X(15,15),IJSSD(15,15)
C     IS,JSD ARE THE SUBSCIPTS OF THE NEW
C     ENTERING BASIC VARIABLE.
C     IJSSD MUST BE=-1OR 0 OR+1.
C     DET.OF THETA MAX.WHICH IS THE SMALLEST OF
C     BASIC X(I,J)FOR IJSSD.GT.0 ONLY.
      THETA=1.0E+38
      DO 15 I=1,M
      DO 15 J=1,N
      IF(IJSSD(I,J).NE.1)GO TO 15
      IF(THETA.LE.X(I,J))GO TO 15
      THETA=X(I,J)
      ILEAVE=I
      JLEAVE=J
C     THESE ARE THE SUBSCRIPTS OF THE LEAVING VARIABLE.
   15 CONTINUE
      X(IS,JSD)=THETA
C     THE ENTERING BASIC VARIABLE.
      IJSSD(IS,JSD)=0
C     CHANGING THE BASIS
      DO 35 I=1,M
      DO 35 J=1,N
      IF(X(I,J).EQ.0)GO TO 35
      IF(IJSSD(I,J))25,35,30
   25 X(I,J)=X(I,J)+THETA
      GO TO 35
   30 X(I,J)=X(I,J)-THETA
      IF(X(I,J).NE.0)GO TO 35
      IF(I.NE.ILEAVE)GO TO 34
      IF(J.EQ.JLEAVE)GO TO 35
   34 X(I,J)=EPS
   35 CONTINUE
      RETURN
      END .
```

Appendix

We present some of the test problems and their results.

Problem. 1:

Minimize,

$$Z = 25x_{11}+10x_{12}+2x_{13}+30x_{14} + 5x_{12} + 15x_{22}+20x_{23}$$

$$10x_{24}+100x_{31} + 65x_{32}+50x_{33} + 2x_{34} .$$

Subject to,

$$x_{11} + x_{12} + x_{13} + x_{14} = 10$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 15$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 20$$

$$x_{11} + x_{21} + x_{31} = 5$$
$$x_{12} + x_{22} + x_{32} = 12$$
$$x_{13} + x_{23} + x_{33} = 13$$
$$x_{14} + x_{24} + x_{34} = 15$$

$x_{ij} \geq 0, \quad i = 1,2,3,, \quad j = 1,2,3,4.$

Problem 2:

Consider the transportation problem with the cost matrix C as shown with the supply and demands given by the vectors

$$a = (40, 70, 60, 30)$$

$$b = (30, 60, 50, 40, 20)$$

$$C = \begin{bmatrix} 5 & 3 & 4 & 7 & 12 \\ 2 & 11 & 8 & 4 & 9 \\ 7 & 8 & 2 & 10 & 12 \\ 11 & 10 & 5 & 13 & 3 \end{bmatrix}$$

Problem 3:

a =(28, 114, 384, 18, 39, 48).

b =(110, 22, 126, 111, 73, 62,69, 26, 13, 19)

$$C = \begin{bmatrix} 1.96 & 1.23 & 2.39 & 2.23 & 3.04 & 4.50 & 5.71 & 8.51 & 9.92 & 9.43 \\ 2.22 & 1.49 & 2.65 & 2.34 & 3.12 & 4.58 & 5.79 & 8.49 & 9.90 & 9.41 \\ 0.65 & 1.32 & 0.38 & 0.48 & 0.85 & 2.33 & 3.24 & 6.04 & 7.45 & 6.96 \\ 3.23 & 3.90 & 3.30 & 3.15 & 3.34 & 0.87 & 0.20 & 3.44 & 4.85 & 4.36 \\ 6.95 & 7.62 & 7.02 & 7.22 & 6.05 & 4.59 & 3.38 & 3.70 & 1.03 & 1.62 \\ 1.99 & 1.55 & 2.21 & 1.72 & 2.55 & 4.17 & 5.38 & 8.18 & 0.59 & 0.10 \end{bmatrix}$$

Problem 4:

$$a = (40, 50, 70, 35, 60, 20)$$

$$b = (20, 30, 40, 80, 60, 30, 15)$$

$$C = \begin{bmatrix} 8 & 4 & 10 & 12 & 7 & 15 & 2 \\ 1 & 7 & 12 & 9 & 11 & 18 & 8 \\ 5 & 4 & 2 & 6 & 1 & 9 & 3 \\ 1 & 1 & 5 & 3 & 3 & 10 & 12] \\ 2 & 4 & 8 & 5 & 7 & 14 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Results: CASE NO. 1

============

25.00	10.00	2.00	30.00	10.00
5.00	15.00	20.00	10.00	15.00
100.00	65.00	50.00	2.00	20.00
5.00	12.00	13.00	15.00	

```
ITERATION NO. 0
----------------
X( 1. 3) =0.100000E  2
X( 2. 1) =0.500000E  1
X( 2. 2) =0.100000E  2
X( 3. 2) =0.200000E  1
X( 3. 3) =0.300000E  1
X( 3. 4) =0.150000E  2
                          O.FUNCTION =0.505000E  3  UNITS OF COST

ITERATION NO. 1
----------------
X( 1. 2) =0.200000E  1
X( 1. 3) =0.800000E  1
X( 2. 1) =0.500000E  1
X( 2. 2) =0.100000E  2
X( 3. 3) =0.500000E  1
X( 3. 4) =0.150000E  2
                          O.FUNCTION =0.491000E  3  UNITS OF COST
                                                ( OPTIMAL SOLUTION )

          IN THIS PROBLEM WE TRANSPORT (     45.000) UNIT WITH TOTAL
COST (0.491000E  3)

                    UPDATE THE TRANSPORTATION PROBLEM
                    ==================================

THE UPDATED PROBLEM NO.(  1)
----------------------------
          WHEN WE INCREASE A( 1) AND B( 4) BY THE SAME AMOUNT
(     5.00) UNIT WE FIND A BETTER SOLUTION.
X( 1. 2) =0.200000E  1
X( 1. 3) =0.130000E  2
X( 2. 1) =0.500000E  1
X( 2. 2) =0.100000E  2
X( 3. 3) =0.100000E -9
X( 3. 4) =0.200000E  2
                          O.FUNCTION =0.261000E  3  UNITS OF COST
                                                [illegible]

          IN THIS PROBLEM WE TRANSPORT (     [illegible]) [illegible]
COST (0.261000E  3)
          THIS PROBLEM CAN NOT BE UPDATED MOR[illegible]
```

CASE NO. 2
===========

5.00	3.00	4.00	7.00	12.00	40.00
2.00	11.00	8.00	4.00	9.00	70.00
7.00	8.00	2.00	10.00	12.00	60.00
11.00	10.00	5.00	13.00	3.00	30.00
30.00	60.00	50.00	40.00	20.00	

ITERATION NO. 0

X(1, 2) =0.400000E 2
X(2, 1) =0.300000E 2
X(2, 4) =0.400000E 2
X(3, 2) =0.100000E 2
X(3, 3) =0.500000E 2
X(4, 2) =0.100000E 2
X(4, 4) =0.100000E -9
X(4, 5) =0.200000E 2

O.FUNCTION =0.680000E 3 UNITS OF COST

ITERATION NO. 1

X(1, 2) =0.400000E 2
X(2, 1) =0.300000E 2
X(2, 4) =0.400000E 2
X(3, 1) =0.100000E -9
X(3, 2) =0.100000E 2
X(3, 3) =0.500000E 2
X(4, 2) =0.100000E 2
X(4, 5) =0.200000E 2

O.FUNCTION =0.680000E 3 UNITS OF COST

(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (200.000) UNIT WITH TOTAL COST (0.680000E 3)

```
                    UPDATE THE TRANSPORTATION PROBLEM
                    =================================
THE UPDATED PROBLEM NO.(  1)
----------------------------
        WHEN WE INCREASE A( 1) AND B( 5) BY THE SAME AMOUNT
(    10.00) UNIT WE FIND A BETTER SOLUTION.
X( 1, 2) =0.500000E  2
X( 2, 1) =0.300000E  2
X( 2, 4) =0.400000E  2
X( 3, 1) =0.100000E -9
X( 3, 2) =0.100000E  2
X( 3, 3) =0.500000E  2
X( 4, 2) =0.100000E -9
X( 4, 5) =0.300000E  2
                    O.FUNCTION =0.640000E  3  UNITS OF COST
                                          ( OPTIMAL SOLUTION )
        IN THIS PROBLEM WE TRANSPORT (    210.000) UNIT WITH TOTAL
COST (0.640000E  3)
THE UPDATED PROBLEM NO.(  2)
----------------------------
        WHEN WE INCREASE A( 1) AND B( 3) BY THE SAME AMOUNT
(    10.00) UNIT WE FIND A BETTER SOLUTION.
X( 1, 2) =0.600000E  2
X( 2, 1) =0.300000E  2
X( 2, 4) =0.400000E  2
X( 3, 1) =0.100000E -9
X( 3, 2) =0.100000E -9
X( 3, 3) =0.600000E  2
X( 4, 2) =0.100000E -9
X( 4, 5) =0.300000E  2
                    O.FUNCTION =0.610000E  3  UNITS OF COST
                                          ( OPTIMAL SOLUTION )
        IN THIS PROBLEM WE TRANSPORT (    220.000) UNIT WITH TOTAL
COST (0.610000E  3)
        THIS PROBLEM CAN NOT BE UPDATED MORE
```

CASE NO. 3

===========

1.96	1.23	2.39	2.23	3.04	4.50	5.71	8.51	9.92	9.43	28
2.22	1.49	2.65	2.34	3.12	4.58	5.79	8.49	9.90	9.41	114
0.65	1.32	0.38	0.48	0.85	2.33	3.24	6.04	7.45	6.96	384
3.23	3.90	3.30	3.15	3.34	0.87	0.20	3.44	4.85	4.36	18
6.95	7.62	7.02	7.22	6.05	4.59	3.38	3.70	1.03	1.62	39
1.99	1.55	2.21	1.72	2.55	4.17	5.38	8.18	0.59	0.10	48
110.00	22.00	126.00	111.00	73.00	62.00	69.00	26.00	13.00	19.00	

ITERATION NO. 0

X(1, 2) =0.220000E 2
X(1, 5) =0.600000E 1
X(2, 5) =0.140000E 2
X(2, 6) =0.620000E 2
X(2, 7) =0.120000E 2
X(2, 8) =0.260000E 2
X(3, 1) =0.110000E 3
X(3, 3) =0.126000E 3
X(3, 4) =0.111000E 3
X(3, 5) =0.370000E 2
X(4, 7) =0.180000E 2
X(5, 7) =0.390000E 2
X(6, 5) =0.160000E 2
X(6, 9) =0.130000E 2
X(6,10) =0.190000E 2

O.FUNCTION =0.105306E 4 UNITS OF COST

ITERATION NO. 10

X(1, 1) =0.280000E 2
X(2, 1) =0.820000E 2
X(2, 2) =0.220000E 2
X(2, 4) =0.100000E 2
X(3, 3) =0.126000E 3
X(3, 4) =0.850000E 2
X(3, 5) =0.730000E 2
X(3, 6) =0.620000E 2
X(3, 7) =0.380000E 2
X(4, 7) =0.180000E 2
X(5, 7) =0.130000E 2
X(5, 8) =0.260000E 2

X(6, 4) =0.160000E 2
X(6, 9) =0.130000E 2
X(6,10) =0.190000E 2
O.FUNCTION =0.892239E 3 UNITS OF COST
(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (631.000) UNIT WITH TOTAL
COST (0.892239E 3)

UPDATE THE TRANSPORTATION PROBLEM
=================================

THE UPDATED PROBLEM NO.(1)

WHEN WE INCREASE A(4) AND B(10) BY THE SAME AMOUNT
(16.00) UNIT WE FIND A BETTER SOLUTION.

X(1, 1) =0.280000E 2
X(2, 1) =0.820000E 2
X(2, 2) =0.220000E 2
X(2, 4) =0.100000E 2
X(3, 3) =0.126000E 3
X(3, 4) =0.101000E 3
X(3, 5) =0.730000E 2
X(3, 6) =0.620000E 2
X(3, 7) =0.220000E 2
X(4, 7) =0.340000E 2
X(5, 7) =0.130000E 2
X(5, 8) =0.260000E 2
X(6, 4) =0.100000E -9
X(6, 9) =0.130000E 2
X(6,10) =0.350000E 2
O.FUNCTION =0.825358E 3 UNITS OF COST
(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (647.000) UNIT WITH TOTAL
COST (0.825358E 3)

THE UPDATED PROBLEM NO.(2)

WHEN WE INCREASE A(4) AND B(2) BY THE SAME AMOUNT
(10.00) UNIT WE FIND A BETTER SOLUTION.

X(1, 1) =0.280000E 2
X(2, 1) =0.820000E 2
X(2, 2) =0.320000E 2
X(2, 4) =0.100000E -9
X(3, 3) =0.126000E 3
X(3, 4) =0.111000E 3

```
X( 3, 5) =0.730000E  2
X( 3, 6) =0.620000E  2
X( 3, 7) =0.120000E  2
X( 4, 7) =0.440000E  2
X( 5, 7) =0.130000E  2
X( 5, 8) =0.260000E  2
X( 6, 4) =0.100000E -9
X( 6, 9) =0.130000E  2
X( 6,10) =0.350000E  2
                              O.FUNCTION =0.791258E  3  UNITS OF COST

                                                      ( OPTIMAL SOLUTION )
        IN THIS PROBLEM WE TRANSPORT (    657.000) UNIT WITH TOTAL
COST (0.791258E  3)
THE UPDATED PROBLEM NO.(  3)
----------------------------
        WHEN WE INCREASE A( 4) AND B( 3) BY THE SAME AMOUNT
(    12.00) UNIT WE FIND A BETTER SOLUTION.
X( 1, 1) =0.280000E  2
X( 2, 1) =0.020000E  2
X( 2, 2) =0.320000E  2
X( 2, 4) =0.100000E -9
X( 3, 3) =0.138000E  3
X( 3, 4) =0.111000E  3
X( 3, 5) =0.730000E  2
X( 3, 6) =0.620000E  2
X( 3, 7) =0.100000E -9
X( 4, 7) =0.560000E  2
X( 5, 7) =0.130000E  2
X( 5, 8) =0.260000E  2
X( 6, 4) =0.100000E -9
X( 6, 9) =0.130000E  2
X( 6,10) =0.350000E  2
                              O.FUNCTION =0.759338E  3  UNITS OF COST
                                                      ( OPTIMAL SOLUTION )
        IN THIS PROBLEM WE TRANSPORT (    669.000) UNIT WITH TOTAL
COST (0.759338E  3)
        THIS PROBLEM CAN NOT BE UPDATED MORE
```

CASE NO. 4

===========

UNDER-PRODUCTION

8.00	4.00	10.00	12.00	7.00	15.00	2.00	40.00
1.00	7.00	12.00	9.00	11.00	18.00	8.00	50.00
5.00	4.00	2.00	6.00	1.00	9.00	3.00	70.00
1.00	1.00	5.00	3.00	3.00	10.00	12.00	35.00
2.00	4.00	8.00	5.00	7.00	14.00	2.00	60.00
0.00	0.00	0.00	0.00	0.00	0.00	0.00	20.00
20.00	30.00	40.00	80.00	60.00	30.00	15.00	

ITERATION NO. 0

X(1, 3) =0.250000E 2
X(1, 7) =0.150000E 2
X(2, 1) =0.100000E -9
X(2, 3) =0.500000E 1
X(2, 4) =0.150000E 2
X(2, 6) =0.300000E 2
X(3, 3) =0.100000E 2
X(3, 5) =0.600000E 2
X(4, 2) =0.300000E 2
X(4, 4) =0.500000E 1
X(5, 4) =0.600000E 2
X(6, 1) =0.200000E 2

O.FUNCTION =0.144000E 4 UNITS OF COST

ITERATION NO. 6

X(1, 2) =0.200000E 2
X(1, 5) =0.500000E 1
X(1, 7) =0.150000E 2
X(2, 1) =0.200000E 2
X(2, 2) =0.100000E 2
X(2, 4) =0.200000E 2
X(3, 3) =0.400000E 2
X(3, 5) =0.300000E 2
X(4, 5) =0.250000E 2
X(4, 6) =0.100000E 2
X(5, 4) =0.600000E 2
X(6, 6) =0.200000E 2

O.FUNCTION =0.100000E 4 UNITS OF COST
(OPTIMAL SO[illegible])

IN THIS PROBLEM WE TRANSPORT (275.000) UNI[illegible] WI[illegible]

COST (0.100000E 4)

UPDATE THE TRANSPORTATION PROBLEM
=================================

THE UPDATED PROBLEM NO.(1)

WHEN WE INCREASE A(6) AND B(1) BY THE SAME AMOUNT
(5.00) UNIT WE FIND A BETTER SOLUTION.

```
X( 1, 2) =0.250000E  2
X( 1, 5) =0.100000E -9
X( 1, 7) =0.150000E  2
X( 2, 1) =0.250000E  2
X( 2, 2) =0.500000E  1
X( 2, 4) =0.200000E  2
X( 3, 3) =0.400000E  2
X( 3, 5) =0.300000E  2
X( 4, 5) =0.300000E  2
X( 4, 6) =0.500000E  1
X( 5, 4) =0.600000E  2
X( 6, 6) =0.250000E  2
```

O.FUNCTION =0.920000E 3 UNITS OF COST
(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (280.000) UNIT WITH TOTAL
COST (0.920000E 3)

THE UPDATED PROBLEM NO.(2)

WHEN WE INCREASE A(6) AND B(5) BY THE SAME AMOUNT
(5.00) UNIT WE FIND A BETTER SOLUTION.

```
X( 1, 2) =0.250000E  2
X( 1, 5) =0.100000E -9
X( 1, 7) =0.150000E  2
X( 2, 1) =0.250000E  2
X( 2, 2) =0.500000E  1
X( 2, 4) =0.200000E  2
X( 3, 3) =0.400000E  2
X( 3, 5) =0.300000E  2
X( 4, 5) =0.350000E  2
X( 4, 6) =0.100000E -9
X( 5, 4) =0.600000E  2
X( 6, 6) =0.300000E  2
```

O.FUNCTION =0.885000E 3 UNITS OF COST
(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (2[illegible]) UNIT WITH TOTAL
COST (0.885000E 3)

THE UPDATED PROBLEM NO.(3)

WHEN WE INCREASE A(5) AND B(1) BY THE SAME AMOUNT (20.00) UNIT WE FIND A BETTER SOLUTION.

X(1, 2) =0.250000E 2
X(1, 5) =0.100000E -9
X(1, 7) =0.150000E 2
X(2, 1) =0.450000E 2
X(2, 2) =0.500000E 1
X(2, 4) =0.100000E -9
X(3, 3) =0.400000E 2
X(3, 5) =0.300000E 2
X(4, 5) =0.350000E 2
X(4, 6) =0.100000E -9
X(5, 4) =0.800000E 2
X(6, 6) =0.300000E 2

O.FUNCTION =0.825000E 3 UNITS OF COST
(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (305.000) UNIT WITH TOTAL COST (0.825000E 3)

THE UPDATED PROBLEM NO.(4)

WHEN WE INCREASE A(1) AND B(1) BY THE SAME AMOUNT (5.00) UNIT WE FIND A BETTER SOLUTION.

X(1, 2) =0.300000E 2
X(1, 5) =0.100000E -9
X(1, 7) =0.150000E 2
X(2, 1) =0.500000E 2
X(2, 2) =0.100000E -9
X(2, 4) =0.100000E -9
X(3, 3) =0.400000E 2
X(3, 5) =0.300000E 2
X(4, 5) =0.350000E 2
X(4, 6) =0.100000E -9
X(5, 4) =0.800000E 2
X(6, 6) =0.300000E 2

O.FUNCTION =0.815000E 3 UNITS OF COST
(OPTIMAL SOLUTION)

IN THIS PROBLEM WE TRANSPORT (310.000) UNIT WITH TOTAL COST (0.815000E 3)

THIS PROBLEM CAN NOT BE UPDATED MORE

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