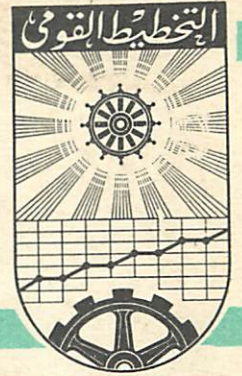


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On a Multiobjective Transportation
Problem.

BY

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Introduction:

For some practical problems, the conventional transportation problem which usually minimizes only the total cost of transportation, became of no benefit to the real situation. For example, high way motor carriers, although considerably cheaper than airfreight, require much longer shipment times, and consequently the freshness and perishability of many grocery products are influenced. Also, for a certain finished goods inventory, the shorter the shipment time the faster is the response of a logistic system to surges in demand. Furthermore, pipeline inventories and associated opportunity costs are directly related to shipment times. Thus, for some problems of real situation, an average shipment time may be used as an objective function beside the total costs. These two objectives, i.e., minimizing total transportation costs and average shipment times are in general conflicting. It is not possible in general to minimize both (or all in case of more than two objective functions) simultaneously. However, there are several different criteria which could be considered and employed to provide alternate solutions to multi-objective problems. One can determine the "efficient" solutions, and consequently the efficient curve, which denote the minimum attainable value of each of the objective functions for different values of the other ones. This idea of determining the efficient curve aids the decision making of the manager in eliminating the inefficient solutions. The decision maker can then subjectively choose that efficient point which mostly suits his company. Another approach to the problem is to find the feasible solution which can be considered the best regarding the optimization of all objective functions. That can be the solution for which the summation of the relative deviations of the different objective functions from their optima is minimum. This turns to--

be the optimal solution of an overall objective function equal to the summation of the different objective function each one weighted by the reciprocal of its optimal value (see, reference (6)). On the other hand if it is possible for the manager to order the objective functions according to their relative periorities, then the problem can be handled as follows: a subset of the overall feasible region which contains optimal solutions to the most important objective function is to be determined. Out of this subset a set of solutions which optimize the second important objective function may be located and so on the process continues for all functions. The most inner subset is considered to be the optimal region for the problem (see, reference (5)). Another alternative is to transform the problem into a program of a single objective function if one of the functions could be considered (by the manager for example) as the most important one over all others. In this cases that function is taken to be the main objective function and all other ones are to be transformed into constraints with lower bounds. The lower bounds represent the required ratios of the optimal values of the objective functions, e.g, it is required at least 90 % of the optimal value of the second objective function, 85% of the fourth one etc. (see, reference (5)).

In this paper, we suggest a method for finding the efficient points of a multiobjective transportation problem when the conflicting objectives of minimizing total costs and shipment times are considered. Corresponding to each efficient point, the method provides the optimal routes, modes of transportation, and the appropriate shipping amounts. The method has been based one the idea of Srinivasan & Thompsons method(see reference 4). In section I we.

present some theoretical results of the multiobjective transportation problem. In section II, a method for finding all optimal basic solution to the standard transportation problem is provided; we sometimes need to identify all optimal solutions (if there exists more than one) to the first or second objective during the process of locating the efficient points. Section III presents an algorithm and a worked example.

I. Theoretical Results For a Multiobjective Transportation Problem

Let

$I = \{ 1, 2, \dots, m \}$, be the set of origins O (rows),

$J = \{ 1, 2, \dots, n \}$, " " " " destinations D (columns)

and
 $K = \{ 1, 2, \dots, P \}$, " " " " modes of transportations.

Let

x_{ijk} be the amount shipped from the origin O_i to destination D_j via mode k ,

c_{ijk} be the unit shipping cost from O_i to D_j via k ,

t_{ijk} be the time of shipping any amount from O_i to D_j via k ,

a_i be the available amount at O_i ($a_i \geq 0$),

and
 b_j be the requirement at D_j ($b_j \geq 0$).

The problem is to generate the efficient solutions to the following multiobjective transportation problem:

Minimize the total cost

$$C(X) = \sum_{i \in I} \sum_{j \in J} \sum_{k \in K} c_{ijk} x_{ijk}$$

and the total time

$$T(X) = \sum_i \sum_j \sum_k t_{ijk} x_{ijk}$$

Subject to

$$\sum_j \sum_k x_{ijk} = a_i \quad i \in I \quad \dots (1)$$

$$\sum_i \sum_k x_{ijk} = b_j \quad j \in J \quad \dots (2)$$

$$x_{ijk} \geq 0, \quad \forall i, j \text{ and } k \quad \dots (3)$$

---(I)

The following conditions are assumed:

- (1) The p modes of transportation are available for all routes. If this is not the case, the unit costs c_{ijk} and times t_{ijk} for the modes not available for a route from O_i to D_j can be set equal to a large positive number M , so that, the optimal solution will not utilize such modes.
- (2) For any route from O_i to D_j there are no two modes k and l with $c_{ijk} \leq c_{ijl} < M$ and $t_{ijk} \leq t_{ijl} < M$. If this is not so, c_{ijl} is set equal to $t_{ijl} = M$.
- (3) A single homogeneous good is considered in the system. If there were multiple goods, the problem may be solved separately for each of the goods.

(4) $\sum_i a_i = \sum_j b_j$. A dummy origin or destination can be added in order

this condition to hold.

A feasible solution X to (I), i.e., satisfying (1) - (3), is called efficient if there does not exist other feasible solution \hat{X} such that

$$\begin{aligned} & C(\hat{X}) \leq C(X) \quad \text{and} \quad T(\hat{X}) < T(X) \\ \text{or} \quad & C(\hat{X}) < C(X) \quad \text{and} \quad T(\hat{X}) \leq T(X). \end{aligned}$$

We now consider the parametric problem $P(\lambda)$:

$$\begin{aligned} & \text{Min} \quad (1 - \lambda)C(X) + \lambda T(X) \quad , \text{ under constraints } \\ & (1)-(3), \text{ where } \quad 0 \leq \lambda \leq 1. \quad \dots\dots\dots \end{aligned} \quad \text{(II)}$$

Problem (II) is equivalent to problem (I) in the sense that: every optimal solution to $P(\lambda)$ is an efficient solution to (I) and conversely, if X is efficient then there exists a scalar λ in the unit interval $(0,1)$ such that X is an optimal solution to $P(\lambda)$. The sufficient condition is Lemma (2) in reference (1).

The necessary condition may be proved as follows:

Let $0 < \lambda < 1$ and assume that an optimal solution X to $P(\lambda)$ is not efficient. Then there exists a feasible solution \hat{X} such that

$$\begin{aligned} & C(\hat{X}) \leq C(X) \quad \text{and} \quad T(\hat{X}) < T(X), \\ \text{or} \quad & C(\hat{X}) < C(X) \quad \text{and} \quad T(\hat{X}) \leq T(X). \end{aligned}$$

Hence $(1 - \lambda) C(\hat{X}) + \lambda T(\hat{X}) < (1 - \lambda) C(X) + \lambda T(X),$

which contradicts the assumption that X was optimal to $P(\lambda)$. The previous argument does not hold when $\lambda = 0$ or $\lambda = 1$. It holds only when there

exists a unique optimal solution to $P(\lambda)$ for both $\lambda = 0$ and $\lambda = 1$. Hence, it is assumed that there exists a unique optimal solution to each of $P(0)$ and $P(1)$.

Let (CXT) denote the two dimensional space with cost C as the abscissa and time T as the ordinate. Let $W(\lambda)$ be the convex set of all optimal solutions to $P(\lambda)$. Thus for each optimal solution $x \in W(\lambda)$, there exists an image $Q(x) = \{ C(x), T(x) \}$ in (CXT) .

Let $Q[W(\lambda)]$ be the set of images corresponding to all solutions in $W(\lambda)$ under this mapping. The following theorem provides a characterization of the set of efficient points.

Theorem 1

(For the proof see Lemma 3 in reference (1)). For each fixed value of λ satisfying $0 \leq \lambda \leq 1$, $Q[W(\lambda)]$ is either a singleton or a compact line segment in (CXT) . In the second case if $Q(x_1)$ and $Q(x_2)$ are the end points of the line segment corresponding to the optimal solutions x_1 and x_2 , then

$$Q(x_1 \delta + (1 - \delta) x_2) = \delta Q(x_1) + (1 - \delta) Q(x_2),$$

for all δ satisfying $0 \leq \delta \leq 1$.

Now, let us introduce the modes of the transportation into the picture.

Let $d_{ij}(\lambda)$ be defined as

$$d_{ij}(\lambda) = \min_{k \in K} \left[(1 - \lambda) c_{ijk} + \lambda t_{ijk} \right]. \dots (4)$$

Let $K_{ij}(\lambda) \subseteq K$ be the set of indices over which (4) attains its

minimum for a particular route (i,j) and for a given value of λ , and the variables y_{ij} be $y_{ij} = \sum_{k \in K_{ij}(\lambda)} x_{ij}$, for a given λ and solution X.

Let us consider the following standard transportation problem $P'(\lambda)$:

$$\min \sum_{i \in I} \sum_{j \in J} d_{ij}(\lambda) y_{ij} \quad \dots (5)$$

subject to

$$\sum_j y_{ij} = a_i, \quad i \in I, \quad \dots (6)$$

$$\sum_i y_{ij} = b_j, \quad j \in J, \quad \dots (7)$$

and

$$y_{ij} \geq 0, \quad \forall i, j. \quad \dots (8)$$

Let

$$c_{ij}^1(\lambda) = \min_{k \in K_{ij}(\lambda)} c_{ijk}, \quad \dots (9)$$

and $k_{ij}^1(\lambda)$ be the index in $K_{ij}(\lambda)$ at which this minimum is attained.

Let $t_{ij}^1(\lambda)$ denote the unit time corresponding to the mode $k_{ij}^1(\lambda)$.

Similarly, let

$$t_{ij}^2(\lambda) = \min_{k \in K_{ij}(\lambda)} t_{ijk}, \quad \dots (10)$$

and $k_{ij}^2(\lambda)$ denote the unique mode in $K_{ij}(\lambda)$ at which this minimum is attained and $c_{ij}^2(\lambda)$ the unit cost corresponding to $k_{ij}^2(\lambda)$. Theorem (2)

below characterizes the optimal solutions to $P(\lambda)$ and Theorem (3) relates the optimum solution of (5) - (9) to that of $P(\lambda)$.

Theorem 2

- (i) For every optimal solution X to $P(\lambda)$, $x_{ijk} = 0$ for $k \in K_{ij}(\lambda)$.
- (ii) Given an optimal solution X to $P(\lambda)$, any other solution X' satisfying:
- $x'_{ijk} \geq 0$ for $k \in K_{ij}(\lambda)$, and
 - $\sum_{k \in K_{ij}(\lambda)} x'_{ijk} = \sum_k x_{ijk} = y_{ij}$ is also optimal to $P(\lambda)$.

Theorem 3

Corresponding to any optimal solution Y to $P'(\lambda)$, (for some given λ), the solution $X_1 = \{x_{ijk}^1\}$ that has the minimum total cost is given by:

$$x_{ijk}^1 = \begin{cases} y_{ij} & k = k_{ij}^1(\lambda) \\ 0 & k \neq k_{ij}^1(\lambda) \end{cases}, \quad \forall i, j \dots \dots (11)$$

The corresponding total cost and time are:

$$C(X^1) = \sum_i \sum_j c_{ij}^1(\lambda) y_{ij} \dots (12)$$

and

$$T(X^1) = \sum_i \sum_j t_{ij}^1(\lambda) y_{ij} \dots (13)$$

The solution X^2 that minimizes the total time is given by

$$x_{ijk}^2 = \begin{cases} y_{ij} & k = k_{ij}^2(\lambda) \\ 0 & k \neq k_{ij}^2(\lambda) \end{cases}, \quad \forall i, j \dots \dots (14)$$

$$C(X^2) = \sum_i \sum_j c_{ij}^2(\lambda) y_{ij} \dots (15)$$

and

$$T(X^2) = \sum_i \sum_j t_{ij}^2(\lambda) y_{ij} \dots (16)$$

From the above theorems, it follows that the set of efficient points (C X T) of problem (I) can be obtained by applying the following basic stages:

- 1) For each chosen value of λ we optimize the problem $p'(\lambda)$.
- 2) Using theorem (3), the optimal solution of $p(\lambda)$ corresponding to that of $p'(\lambda)$ is constructed.
- 3) If for any λ the optimal solution is not unique, we choose among the alternate optimal solutions a limit-cost solution X^c and a limit-time solution X^t such that $C(X^c) \leq C(X)$ and $T(X^t) \leq T(X)$, where X is any optimal solution to $p(\lambda)$.
- 4) By theorem 1, the set of efficient points is the line segment connecting the limit-cost point $[C(X^c), T(X^c)]$ and the limit-time point $[C(X^t), T(X^t)]$.

(If the optimal solution is unique for any λ then the limit-cost and limit-time points coincide).

It is well known that any optimal solution to a linear programming problem is a convex combination of its basic optimal solutions. The following theorem uses this result to identify the limit-cost and limit-time solutions for any λ . Using this theorem we can implement the previous stage 3.

Theorem 4*

Let Y_1, Y_2, \dots, Y_g be the basic optimal solutions to $p'(\lambda)$. Let $X_1^1, X_2^1, \dots, X_g^1$ and $X_1^2, X_2^2, \dots, X_g^2$ be the solutions constructed

* For the proofs of theorems (2)-(4) see reference (4).

corresponding to Y_1, \dots, Y_g using theorem (3). Then the limit-cost solution X_1 to $p(\lambda)$ is given by that solution which minimizes $C(X_i^1)$, $i=1, \dots, g$, and the limit-time solution X^2 is given by that solution which minimizes $T(X_i^2)$, $i = 1, \dots, g$.

II- A Method for Generating All the Alternate Basic Optimal Solutions to the Transportation Problem

Let us assume that the transportation problem; min.

$$Z = \sum_i \sum_j C_{ij} x_{ij} \text{ subject to } \sum_i x_{ij} = b_i, \sum_j x_{ij} = a_j,$$

$i = 1, \dots, m, \quad j = 1, \dots, n, \quad \text{and } x_{ij} \geq 0$, has more than one

optimal solution. This can be recognized from the final optimal transportation tableau, when the coefficients of some of the nonbasic variables have

the value zero. An indicator α of a basic optimal solution Y to the previous transportation problem is the set $(i_1, i_2, \dots, i_{m+n-1}) \subset I \cup J$, where

$i_1, i_2, \dots, i_{m+n-1}$ are the indices of the basic variables. Two indicators

α_1, α_2 are said to be neighbouring indicators if all except one of the

elements of α_1 and α_2 are the same. The neighbouring set of an indicator α consists of all neighbouring indicators to α and will be denoted by $G(\alpha)$.

Let P denote the set of all indicators of the optimal basic solutions.

The method is based on the use of two sets. The first is $S^r \subset P$, we call it

the served set, and the second is $W^r \subset P$, we call it the waiting set. We

start the process with an optimal basic solution Y_0 . We store the indicator

α_0 associated with Y_0 in the set $S^r, r=1$, i.e., $S^1=0$. We analyze the optimal

transportation tableau corresponding to α_0 to specify all the nonbasic variables

of zero coefficients. If more than one neighbouring indicator of α_0 will

be created from the current tableau, we store all of them except α_0 in the set $W^{r,r=1}$, i.e., $W^1 = G(\alpha_0) - \alpha_0$. If only one indicator is specified, say α we calculate its corresponding solution and put α in $S^{r,r=2}$, i.e., $S^2 = \alpha_0 \cup \alpha$. We pick the last element stored in W^1 , say α_1 if more than one indicator have been stored in W^1 , and calculate the associated basic optimal solution. We examine the optimal transportation tableau corresponding to α_1 in order to create the new indicators (if there is any) neighbouring to α_1 . We update S^2 and W^2 by adding the new neighbouring indicators of α_1 to W^2 and removing α_1 from W^2 and adding it to S^2 so we get

$$S^2 = \alpha_0 \cup \alpha_1 \quad \text{and} \quad W^2 = W^1 \cup G(\alpha_1) - S^2.$$

We pick the last indicator stored in S^2 and repeat the same process. At the g -th stage the waiting set W^g and served set S^g will be:

$$S^g = S^{g-1} \cup \alpha_{g-1} \quad \text{and} \quad W^g = W^{g-1} \cup G(\alpha_{g-1}) - S^g.$$

The process will terminate finitely when $W^r = \emptyset$. We now prove that all optimal basic solutions will have been found finitely when $W^r = \emptyset$. The proof runs as follows:

From the construction of S^g and W^g we have $S^g \cap W^g = \emptyset$. Since the number of basic optimal solutions is finite ($\frac{(nm + n + m)!}{nm! (n + m)!}$ is an upper bound) and any indicator in W^g will sooner or later leave W^g and enter S^g then after a finite number of iterations W^r will be empty. The only statement to be proved^{is} that $W^r = \emptyset$ implies $S^r = P$.

Let α_0 be an element of P . Since we start with an optimal basic solution then $S^r \neq \emptyset$. Let α_1 be an element of $S^r \subset P$. Then we have

a finite sequence of neighbouring indicators

$$\alpha_1, \alpha_2, \dots, \alpha_k = \alpha_0,$$

such that

$$\alpha_{j+1} \in G(\alpha_j), \quad j = 1, 2, \dots, k-1.$$

The existence of this sequence follows from the fact that any optimal basic solution to the transportation problem can be obtained from any other optimal basic solution by a finite number of iterations; this follows from the fact that the graph of the feasible region of any linear programming problem is connected. If some $\alpha_j \in S^r$, then it had entered S^r at some iteration $g_i \leq r$, say. Thus either α_{j+1} is an element of S^{g_i} , and hence $\alpha_{j+1} \in S^r$, or α_{j+1} is not an element of S^{g_i} and $\alpha_{j+1} \in G(\alpha_j)$ which implies $\alpha_{j+1} \in W^{g_i+1}$. Since $W^r = \emptyset$, then α_{j+1} must have left W^r at some iteration between the $(g_i + 1)$ -th and the r -th stage and entered the set S^{g_i+1} so that $\alpha_{j+1} \in S^r$. Thus in any case $\alpha_j \in S^r$ implies $\alpha_{j+1} \in S^r$. Since $\alpha_1 \in S^r$ we have by induction $\alpha_k = \alpha_0 \in S^r$. Therefore $P \subseteq S^r$. But $S^r \subseteq P$. whence $P = S^r$.

The method presented in this section will be used to find the alternate basic optimal solutions required by theorem (4).

III Statement of The Algorithm And a Worked Example.

We combine the results of the previous two sections to provide the algorithm below. Applying the following steps, the set of efficient points of problem (I) can be located, and the optimal transportation routes, modes and shipping amounts can be determined.

Step 1. Let $\lambda = 0$. Determine $d_{ij}(0)$ and $K_{ij}(0)$ by using equation (4).*

Using equation (9) - (10), find

$$k_{ij}^1(0), k_{ij}^2(0), c_{ij}^1(0), t_{ij}^1(0), c_{ij}^2(0), \text{ and } t_{ij}^2(0),$$

for all $i \in I$ & $j \in J$. Solve the transportation problem $P^{\wedge}(0)$, i.e.,

(5) - (8), and find the basic optimal solution Y^{\wedge} .

Corresponding to Y^{\wedge} find the limit - cost solution X^1 by equation

(11).

Find the limit-cost point $[C(X^1), T(X^1)]$ by (12) - (13). The efficient point in (CXT) space corresponding to $\lambda = 0$ is given by $[C(X^1), T(X^1)]$

(Note that Y^{\wedge} is unique by assumption, thus the limit-cost and time points coincide for $\lambda = 0$).

Step 2. Choose different values for λ in the interval $(0,1)$. For each chosen value of λ , apply the following:

(i) Determine $K_{ij}(\lambda)$ using equation (4) and evaluate $k_{ij}^1(\lambda), k_{ij}^2(\lambda)$, and the corresponding $c_{ij}^1(\lambda), t_{ij}^1(\lambda), c_{ij}^2(\lambda)$ and $t_{ij}^2(\lambda)$ from from equations (9)-(10) for all $i \neq j$.

(ii) Solve the problem $P^{\wedge}(\lambda)$. There are two different cases:

a) If the basic optimal solution Y^{\wedge} to $P^{\wedge}(\lambda)$ is unique, then find the limit-cost solution X^1 using (11) and the limit-cost point $[C(X^1), T(X^1)]$ using (12)-(13). Similarly determine the limit-time solution X^2 using (14) and the limit-time point $[C(X^2), T(X^2)]$ using (14)-(15).

* Since it is assumed that there are no two modes with the same C_{ijk} , then the set $K_{ij}(0)$ is a singleton.

Label the line segment connecting the limit-cost and limit-time points as the set of efficient points corresponding to λ (theorem (1)).

b) If the problem $P(\lambda)$ has alternate optimal solutions^{*}, then use the method of section (II) to get all basic optimal solutions Y_i , $i=1,2,\dots,g$.

Corresponding to every basic optimal solution Y_i , determine $C(X_i^1)$ using equation (12). For the Y which minimizes $C(X_i^1)$ determine the limit-cost solution X^1 by equation (11) and the limit-cost point $[C(X^1), T(X^1)]$ by (11) - (13). Corresponding to every basic optimal solution determine $T(X_i^2)$ using equation (14). For the Y which minimizes $T(X_i^2)$ determine the limit-time solution X^2 by (14) and the limit-time point $[C(X^2), T(X^2)]$ by (15) - (16).

Label the line segment connecting the limit-cost and time points as the set of efficient points corresponding to λ .

Step 3 Once an efficient point is chosen from the efficient curve, find the corresponding value λ^* . Determine δ^* such that

$$(C,T) = \delta^* [C(X^1), T(X^1)] + (1 - \delta^*) [C(X^2), T(X^2)]$$

where the terms in square brackets are the limit-cost and time points for λ^* .

The optimal solution

$$X^* = \delta^* X^1 + (1 - \delta^*) X^2, \text{ determines the optimal routes, modes,}$$

and the shipping amounts associated with (C,T) .

* This will be recognized when some of the nonbasic variables have zero coefficients in the final optimal tableau.

A Worked Example

Let us have the following data for 3 ware houses (W_i), 3 markets (M_j) and 3 modes of transportation.

6	4	8
8	5	7
20	10	16
11	9	13
2	6	18
3	5	10

Mode No.1
Highway

10	6	12
6	3	3
28	14	22
7	5	7
4	9	30
2	4	4

Mode No.2
Railroad

16	11	16
3	2	1
48	19	24
3	4	6
9	10	40
1	3	2

Mode No.3
Airfreight

For each of the three modes $k=1,2,3$, the top entry in cell (i,j) denotes the cost C_{ijk} of shipping one unit of the good from W_i to M_j Via mode k . The bottom entry in (i,j) denotes the time t_{ijk} for shipping any amount of the good from W_i to M_j Via mode k . The available amount for W_1, W_2 and W_3 are 50, 70 and 80 units, respectively; the requirements b_j for M_1, M_2 , and M_3 are 110, 60, and 30 units respectively.

We set $\lambda = 0$. Then

$$d_{ij}(0) = \min_{k=1,2,3} C_{ijk} = C_{ij} \quad \text{For all } i \text{ and } j \quad K_{ij}(0) = \{1\}$$

$$K_{ij}^1(0) = K_{ij}^2(0) = \{1\}, \text{ also, } C_{ij}^1(0) = C_{ij}^2(0) = C_{ij} \text{ and } t_{ij}^1(0) = t_{ij}^2(0) = t_{ij} \text{ for all } i \text{ and } j.$$

The transportation problem $P^1(0)$ and its optimal basic solution obtained by MODI method (see, reference (2)), are shown below. The circles denote

the basis cells with the amounts y_{ij} shown inside the circles.

$$\sum_i \sum_j C_{ij} y_{ij} = 6y_{11} + 4y_{12} + 8y_{13} + 20y_{21} + 10y_{22} + 16y_{23} + 2y_{31} + 6y_{32} + 18y_{33}$$

subject to

$$\sum_{j=1}^3 y_{ij} = a_i, \quad i=1,2,3$$

$$\sum_{i=1}^3 y_{ij} = b_j, \quad j=1,2,3,$$

$$y_{ij} \geq 0.$$

	M_1	M_2	M_3
W_1	6 30	4	8 20
W_2	20	10 60	16 10
W_3	2 80	6	18

From equations (12) and (13) we get

$$C(X^1) = \sum_i \sum_j C_{ij}^1(0) y_{ij} = \sum_i \sum_j C_{ij} y_{ij}$$

$$= 6 \times 30 + 8 \times 20 + 10 \times 60 + 16 \times 10 + 2 \times 80 = 1260, \text{ and}$$

$$T(X^1) = \sum_i \sum_j t_{ij}^1(0) y_{ij} = \sum_i \sum_j t_{ij} y_{ij} =$$

$$30 \times 8 + 20 \times 7 + 60 \times 9 + 10 \times 13 + 80 \times 3 = 1290.$$

For $\lambda = 0$ the limit - cost and limit - time points coincide, then this efficient point is shown as the point A in Figure (1).

Now, we choose the value $\frac{1}{2}$ for λ . Then the coefficients of the problem

$p'(1/2)$ are:

$$d_{ij}(\frac{1}{2}) = \min_{k=1,2,3} \left\{ \frac{1}{2} c_{ij} + \frac{1}{2} t_{ij} \right\}, \text{ for } i \in I, j \in J.$$

Using equations (4), (9), and (10), we calculated $K_i(\frac{1}{2})$,

$R_{ij}^1(\frac{1}{2})$, $R_{ij}^2(\frac{1}{2})$, $C_{ij}^1(\frac{1}{2})$, $C_{ij}^2(\frac{1}{2})$, $t_{ij}^1(\frac{1}{2})$, and $t_{ij}^2(\frac{1}{2})$ as shown in Table

(1) below.

The problem $p'(\frac{1}{2})$ takes the form:

$$\begin{aligned} \min \quad & 7y_{11} + 9/2y_{12} + 15/2y_{13} + 31/2y_{21} + 19/2y_{22} + \\ & 29/2y_{23} + 5/2y_{31} + 11/2y_{32} + 14y_{33} \end{aligned}$$

$$\text{Subject to } \sum_{j \in I} y_{ij} = a_i, \quad i=1,2,3$$

$$\sum_{i=1}^3 y_{ij} = b_j, \quad j=1,2,3,$$

and $i=1$

$$y_{ij} \geq 0.$$

Using MODI method, we get the optimal basic solution: $y_{11} = 30$, $y_{13} = 20$,

$$y_{22} = 60, \quad y_{23} = 10, \quad y_{31} = 80.$$

The optimal solution is unique, because the coefficients of the nonbasic variables are all strictly negative.

From (12) and (13), we get:

$$C(X^1) = 6 \times 30 + 20 \times 8 + 60 \times 10 + 10 \times 16 + 2 \times 80 = 1260$$

$$\& T(X^1) = 8 \times 30 + 20 \times 7 + 60 \times 9 + 10 \times 13 + 80 \times 3 = 1280.$$

Thus this limit-cost point $[C(X^1), T(X^1)]$ is the same as the point A in Figure

(1) obtained earlier.

The limit-time point, using (14) and (15), is:

Table (1)

(i,j)	$\frac{1}{2} c_{ijk} + \frac{1}{2} t_{ijk}$	$d_{ij}(\frac{1}{2})$	$k_{ij}(\frac{1}{2})$	$k_{ij}^1(\frac{1}{2})$	$c_{ij}^1(\frac{1}{2})$	$t_{ij}^1(\frac{1}{2})$	$k_{ij}^2(\frac{1}{2})$	$c_{ij}^2(\frac{1}{2})$	$t_{ij}^2(\frac{1}{2})$
(1,1)	7,8,19/2	7	1	1	6	8	1	6	8
(1,2)	9/2,9/2,13/2	9/2	{1,2}	1	4	5	2	6	3
(1,3)	15/2,15/2,17/2	15/2	{1,2}	1	8	7	2	12	3
(2,1)	31/2,35/2,51/2	31/2	{1}	1	20	11	1	20	11
(2,2)	19/2,19/2,23/2	19/2	{1,2}	1	10	9	2	14	5
(2,3)	29/2,29/2,15	29/2	{1,2}	1	16	13	2	22	7
(3,1)	5/2, 3,5	5/2	{1}	1	2	3	1	2	3
(3,2)	11/2,13/2,13/2	11/2	{1}	1	6	5	1	6	5
(3,3)	14,17,21	14	{1}	1	18	10	1	18	10

Computation of $k_{ij}(\frac{1}{2})$, $k_{ij}^1(\frac{1}{2})$, and $k_{ij}^2(\frac{1}{2})$.

$$C(x^2) = \sum_i \sum_j c_{ij}^2 \left(\frac{1}{2}\right) y_{ij} = 1640 \text{ and}$$

$$T(x^2) = \sum_i \sum_j t_{ij}^2 \left(\frac{1}{2}\right) y_{ij} = 910.$$

Then the limit-time point for $\lambda = \frac{1}{2}$ is the point B = (1640, 910) in Fig.

(1). Therefore, using theorem (1), the line segment connecting A and B in

Fig.(1) locates the efficient points for $\lambda = \frac{1}{2}$. Since the point A =

(1260, 1290) was resulted from solving both the problems $p'(0)$ and $p'(\frac{1}{2})$

then A is the efficient point for $0 \leq \lambda \leq \frac{1}{2}$.

Next, we choose $\lambda = \frac{2}{3}$. For this value of λ we find that the limit-cost point and limit-time point coincide with the point B in Fig. (1). We choose

$\lambda = \frac{2}{3}$. The computed values of $d_{ij}(\frac{2}{3}), K_{ij}(\frac{2}{3}), K_{ij}^1(\frac{2}{3}), \dots, t_{ij}^2(\frac{2}{3})$ are shown in Table (2).

Table (2)

(i,j)	$\frac{1}{3} c_{ijk} + \frac{2}{3} t_{ij,k}$	$d_{ij}(\frac{2}{3})$	$K_{ij}(\frac{2}{3})$	$k_{ij}^1(\frac{2}{3})$	$c_{ij}^1(\frac{2}{3})$	t_{ij}^1	$k_{ij}^2(\frac{2}{3})$	$c_{ij}^2(\frac{2}{3})$	t_{ij}^2	$(\frac{2}{3})$
(1,1)	22/3, ..., 22/3	22/3	{1,2,3}	1	6	8	3	16	3	
(1,2)	14/3, 12/3, 15/3	12/3	{2}	2	6	3	2	6	3	
(1,3)	22/3, 18/3, 18/3	18/3	{2,3}	2	12	3	3	16	1	
(2,1)	42/3, 42/3, 54/3	42/3	{1,2}	1	20	11	2	28	7	
(2,2)	28/3, 24/3, 27/3	24/3	{2}	2	14	5	2	14	5	
(2,3)	42/3, 36/3, 36/3	36/3	{2,3}	2	22	7	3	24	6	
(3,1)	8/3, 8/3, 11/3	8/3	{1,2}	1	2	3	2	4	2	
(3,2)	16/3, 17/3, 16/3	16/3	{1,3}	1	6	5	3	10	3	
(3,3)	38/3, 38/3, 44/3	38/3	{1,2}	1	18	10	2	30	4	

Computation of $k_{ij}(\frac{2}{3})$, $k_{ij}^1(\frac{2}{3})$ and $k_{ij}^2(\frac{2}{3})$.

Now solving the problem $p'(2/3)$:

$$\min \sum_i \sum_j d_{ij} (2/3) y_{ij}, \text{ subject to } \sum_i y_{ij} = b_j,$$

$j=1,2,3, \sum_j y_{ij} = a_i, i=1,2,3,$ and $y_{ij} \geq 0,$ we get the following unique optimal basic solution:

30	-6/3	20
-2/3	60	10
80	-24/3	-34/3

The values appears in the nonbasic cells are the coefficients of the nonbasic variables.

Table (3)

The limit-cost point for $\lambda = 2/3$ is the point (1640,910) which is the same as the point B. However, the limit-time point calculated by (14) & (15) is $(C(X^2), T(X^2)) = (2200,630)$. This point is shown as point C in Fig. (1). Hence, the point B is the efficient point for $\frac{1}{2} \leq \lambda \leq 2/3,$ and the line segment BC represents all the efficient points for $\lambda = 2/3.$

$$\text{Let } \lambda=1. \text{ Then } d_{ij}(1) = \min_{k=1,2,3} t_{ijk} = t_{ij3} \text{ and}$$

$$K_{ij}(1) = \{3\} = k_{ij}^2 = k_{ij}^1, \quad C_{ij}^1(1) = C_{ij}^2(1) = C_{ij3}, \text{ and } t_{ij}^1(1) = t_{ij}^2(1) = t_{ij3}$$

The optimal basic solution to $p'(1)$ is

3	2	20	30
3	4	40	6
1	3	2	

using equations (12) and (19), we get

$$c(X^1) = \sum_i \sum_j c_{ij}^1 (1) y_{ij} = 3620 \text{ and}$$

$$T(X^1) = \sum_i \sum_j t_{ij}^1 (1) y_{ij} = 400.$$

The efficient point $(C, T) = (3620, 400)$ is shown in Fig. (1) as the point N. To connect the point C with N we search for the values of λ 's which have continuous coincidence of limit-cost points with limit-time points of the preceding ones. For example, we found that the limit-time point c for $\lambda=2/3$ coincides with the limit-cost point D for $\lambda = 4/5$, the limit-time point D = (2240, 620) coincides with the limit-cost point E = (340, 440) for $\lambda = 5/6$, and the limit-time point E coincides with the limit-cost point N=(3620,400) , as shown in Fig(1).

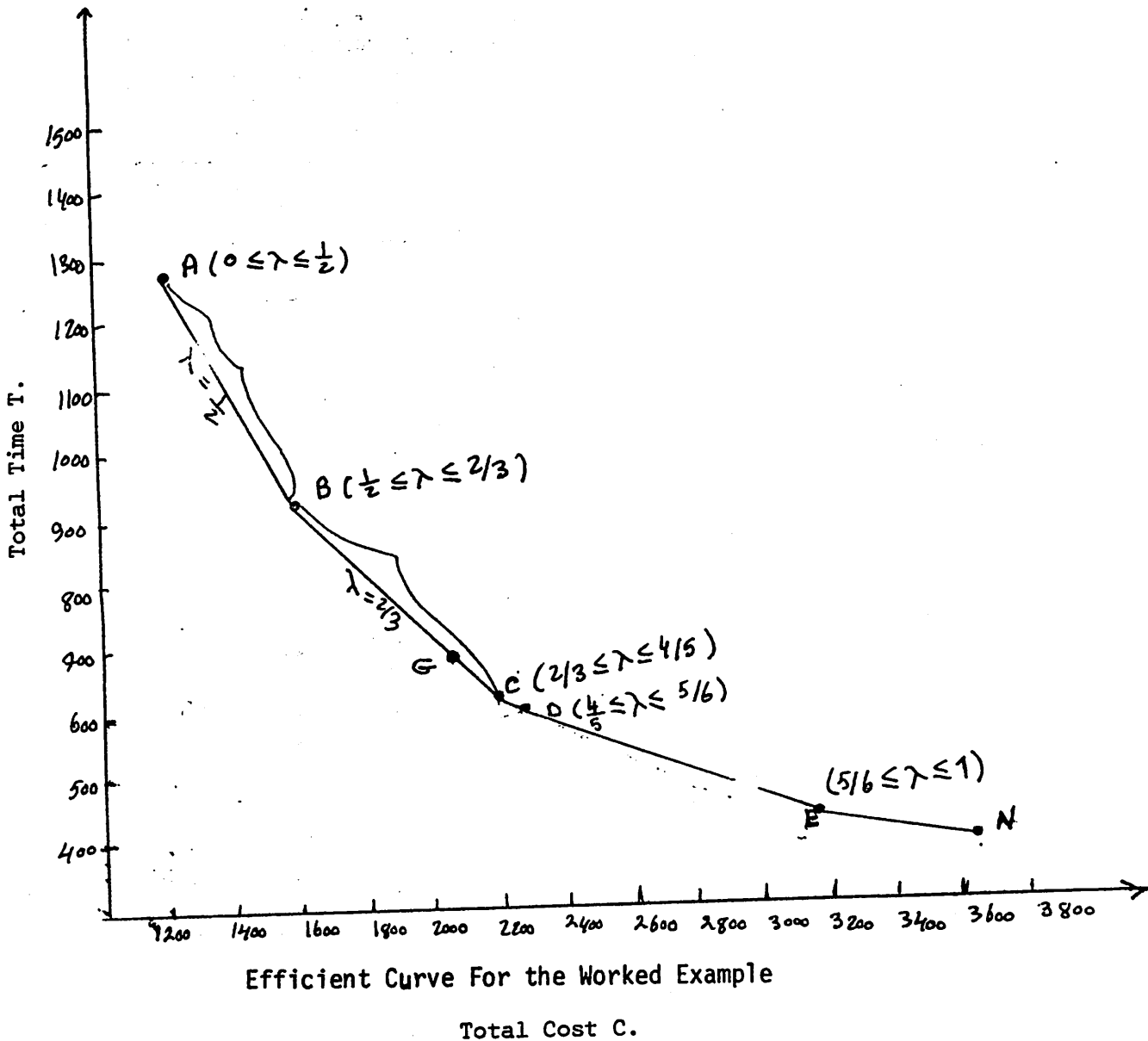
Now, the line segments connecting the efficient points in Fig.(1) represent the efficient points of problem $p(\lambda)$.

Now the task of the manager has been simplified since he has to choose only a point from the efficient points along the efficient curve. Suppose that he subjectively chooses the point c. Then the optimal shipping routes and optimal amounts are as shown in Table (3). The optimal modes of shipment of c are given by the k_{ij}^2 (2/3)-column of Table (2), e.g., $x_{111}=x_{112} = 0$ & $x_{113}=30$, $x_{221} = x_{223}=0$ & $x_{222} = 60$. The optimal solution of the point c uses only rail-road and airfreight. If the point B was chosen, the optimal shipping routes and amounts remain the same but the optimal modes are those given by the column k_{ij}^1 (2/3) in Table (2); only high-way and railroad may be used. If the manager were to choose the efficient point G with $C=2060$ and $T=700$, i.e., G is 3/4 the way between B and C, then the corresponding optimal solution would be

$$X^G = \frac{3}{4} X^C + \frac{1}{4} X^B$$

where X^C and X^B denote the solutions corresponding to the efficient points C and B, for example, for $(i,j) = (1,1)$, $X_{111}^G = 7.5$, $X_{112}^G = 0$, $X_{113}^G = 22.5..$

Figure 1

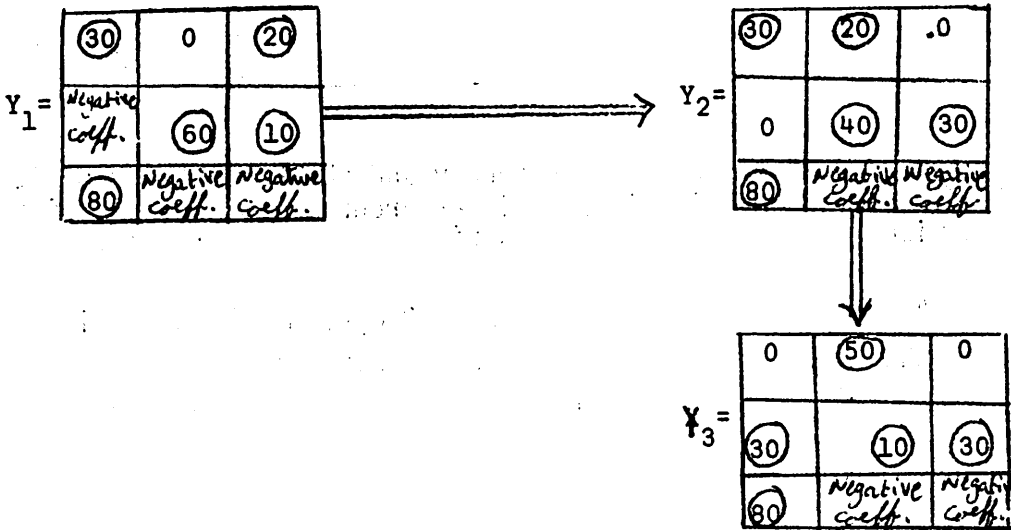


To see how the process be carried on when the problem $p'(\lambda)$ has more than one basic optimal solutions for certain value of λ , let us assume that $p'(2/3)$ has

more than one optimal solutions and the final optimal transportation tableau for p' (2/3) were:

(30)	0	(20)
Negative coeff.	(60)	(10)
(80)	Negative coeff.	Negative coeff.

where the coefficient of X_{12} were zero. If the coefficients $d_{13}^{(2/3)}$ and $d_{22}^{(2/3)}$ were 8 and $34/3$ instead of $18/3$ and $42/3$ respectively, and if we use the method of section (II), we get the following basic optimal solutions X_1, X_2 , and X_3 to $VP'(2/3)$:



Using equation (12), we obtain: $C(X_1^1) = 1640$, $C(X_2^1) = 1680$, and $C(X_3^1) = 1860$.

Using equation (16), we have

$$T(X_1^2) = 630, T(X_2^2) = 690, \text{ and } T(X_3^2) = 750.$$

Hence, we have to choose $[C(X_1^1), T(X_1^1)] = (1640, 910)$

as the limit-cost point since $C(X_1^1) = \min_{i=1,2,3} \{C(X_i^1)\}$.

The limit-time point were to be $[C(X_1^2), T(X_1^2)] = (2200, 630)$, for $T(X_1^2) = \min_{i=1,2,3} \{T(X_i^2)\}$.

These two points occasionally coincide with the points B and C.

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