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The Generation of all Efficient Extreme points for a multiple objective Linear Program

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INTRODUCTION

It has frequently been argued that the traditional approximation of multiple goals of the decision models by a single criterion is either inappropriate or incorrect. In reality, decisicon situation is characterized by a series of conflicting goals, and it might be an impossible task to tie all goals into a single unifying trade-off function. Recently, the search for a discovery of concepts, theories, tools, and solving algorithms applicable to multiabjective linear programs has been continuing in order to serve the decision-making processes.

The linear programming problem involving multiple objective functions induces substituation of a single optimal solution by a set of suboptimizations. The suboptimization situation could be the best possible values, under the given conditions, for the considered objective functions, or could be the full optimization of one (or more) objective in the expense of a lower degree of attainment of the other objectives, or other considerations. It might be worthwhile to state the following quotation:

John Von Neumann " ... This multiple objective situation is certainly no maximum problem, but a peculiar and disconcerting mixture of several confticting maximum problems... This kind of problem is no - where dealt with in classical mathematics. We emphasize at the risk of being pedantic that this is no conditional maximum problem, no problem of the calculus of functional analysis, etc. ..."

The efficient solution is considered as a technical interpretation of the multiple objective situation. In recent years, the theory of vector function maximization problem has been developed, especially in the direction of algorithmic developments. As a consequence, the charact erization and determination of the set of efficient solutions has become one of the main targets. Though the interest in the description of the efficient set has increased substantially, no satisfactory algorithms for generating all efficient solutions have been found yet. Some of the algorithms for locating efficient solutions are presented in (Zeleny,6), and (Isermann, 3).

We give here a computational algorithm with some new features for generating all efficient extreme points for a multiple objective linear program. The algorithm seems to provide a computationally effective method.

Notations, Definitions, and Basic Theorems

Let the linear multiple objective programming problem be in the form

Maximize the vector - volued

$F(x) = (c^1 x, c^2 x, \ldots, c^r X)$

Subject to

$$Ax = d , \qquad \ldots (I)$$

and $x \geq o$,

where A is mxn coefficient matrix of rank m, n > m, d is a requirement m - column vector, and x is the m - column vector of variables, The components of F (x) are the objectives that are to be maximized over the convex polyhedron $\underline{X} = \{ x \mid Ax = d, x \geqslant 0 \}$.

Let B denote the basic matrix of order m x m, the J-th column vector of A will be denoted by the small letter a_j , Let $X = (x_1, x_2, \ldots, x_k)$ and $y = (y_1, y_2, \ldots, y_k)$ be two vectors, then

(i) $x > Y \Leftrightarrow x_i > y_i$, $i = 1, 2, \ldots, k$

(ii) $x \geqslant Y \Leftrightarrow x_i \geqslant y_i$, $i = 1, 2, \ldots, k$ and $x \neq y$

(iii) $x = Y \Leftrightarrow x_i = y_i$, $i = 1, 2, \ldots, k$.

Definition 1

The point $\overset{o}{x} \in \underline{X}$ is called efficient if there is no other $x \in \underline{X}$ such that $F(x) \geqslant F(\overset{o}{x})$. That is, there is no $\sum_{j=1}^{n} c^i_{j_d} x_{j_d} > \sum_{j=1}^{n} c^i_d \overset{o}{x}_j$ for any $i \in \{1, 2, \ldots, r\}$.

Definition 2

We call $x \in \underline{X}$ an alternative efficient solution to (I) if $F(x) = F(\overset{o}{x})$, $x \neq \overset{o}{x}$ and $\overset{o}{x}$ is efficient.

Definition 3

The efficient basic solution $\overset{o}{x}$ is called degenerate if one or more of the basic variables of $\overset{o}{x}$ has the value zero.

Let x be an extreme point (a basic feasible solution) of $\overline{X}$. Then corres ponding to each nonbasic column a_j of A there exists a vector.

$$Z_j = (Z_j^1, Z_j^2, \ldots, Z_j^r), \text{ where}$$

$Z_j^i = C_B^i B^{-1} a_j - c_j^i$, $i = 1, 2, \ldots, r$. $c_B^i = (C_{B_1}^i, \ldots, C_{B_m}^i)$ is the prices of the basic variables of x in the i - th objective function, For the basic columns $Z_j^i = 0$, $i = 1, \ldots, r$. In addition, there associates to x a vector of values of the r - objective functions:

$$F = (f^1, f^2, \ldots, f^r),$$

where $f^i = \sum_{j=1}^{n} C_j^i x_j$, $i = 1, \ldots, r$.

Let E denote the set of all efficient extreme points, and let θ be defined a s

$$\theta_j = \min_{i=1, \ldots, m} \left\{ \frac{X_{Bi}}{y_{ij}} , y_{ij} > 0 \right\}, \text{where}$$

$Y_j = (Y_{1j}, \ldots, Y_{mj}) = B^{-1} a_j$, a_j is a nonbasic column, and $x_{B1}, x_{B2}, \ldots, X_{Bm}$ are the basic variables of x.

<u>Theorem 1</u>

Let $\theta > 0$ (i). Then the extreme point $\overset{0}{x} \notin E$ if $Z_j \leq 0$, for any nonbasic column a_j (see zelény6)

(ii) If $\overset{0}{x}$ is efficient and $Z_j = 0$ for any nonbasic column, then introducing the j - th column a_j into the basis will lead to an alternative efficient point $\hat{x}$.

Proof If $z_j = 0$, then the new values of the objective functions $\hat{F} = \overset{\bullet}{F} - \theta_j Z_j = \overset{b}{F}$, and since $\theta_j \neq 0$ then $\hat{x} \neq \overset{o}{x}$.

Theorem 2 (Zeleny, 6)

If any objective function f_i, $i = 1, \ldots, r$. is at its unique maximum value at the extreme point $\overset{o}{x}$, then $\overset{o}{x} \in E$. In case a function has atternative optimal solutions at $\overset{o}{x}$, then some of the alternate solutions may be noneffi-cient.

Theorem 3 (Zeleny, 6)

Solve the problem:

Maximize $V = \sum_{i=1}^{r} V_i$

Subject to $Ax = d$

$$C_i^i x - V_i = C^i \overline{x} \qquad \ldots) \text{ (II)}$$

$$i = 1, 2, \ldots, r,$$

$$X \geqslant o \text{ and } V_i \geqslant o.$$

Then $\overline{X} \notin E$ if and only if Max $V > 0$ and $\overline{X} \in E$ if and only if Max $V = 0$.

Theorem 3 Can be used to check the efficiency of an extreme point of X. To illustirate the application of this theorem, we analyize the simplex table-aux associated with the constructed problem (II). Let use define the following symbols:

C - (r x n) - matrix of coefficients of the r objectives.

B_k - (mxm) basic matrix at the k - th simplex step.

$\overline{X}$ - n - vector of variables

C_B — (r x m) matrix, of the prices in the objective functions, correspoi- to basic vectors in B_k.

I - identity matrix (of proper order).

0 - Zero matrix (" ").

For the original problem (I), the simplex tableau corresponding to the extreme point $\overline{x}$ is given by:

Table (1)

(1)	$B_k^{-1}A$	L	$B_k^{-1}d$
(2)	$C_B B_k^{-1} A - C$	$C_B B_k^{-1}$	$C_B B_k^{-1} d$

Part (2) consists of r rows each corresponds to one of the objective functions. $C_B B_k^{-1} d$ are the values of the objective functions at $\overline{X}$, i.e

$C_B B_k^{-1} d = c\overline{X}$.

For the constructed problem (II) with the appended constraints $C^i X - v_i + w_i = C^i \overline{X}$, where w_i are the artificial variables added, the initial simplex tableau takes the form:

Table (2)

A	$I_{m \times m}$	$0_{m \times r}$	$0_{m \times r}$	d
C	$0_{r \times m}$	$-I_{r \times r}$	$I_{r \times r}$	$C\overline{X}$
-C	$0_{r \times m}$	$0_{r \times r}$	$0_{r \times r}$	$0_{r \times 1}$

corresponding to $\bar{x}$ and its basis B_k, table (2) has the form

Table (3)

$$
\begin{array}{l}
(3)\\
(4)\\
(5)
\end{array}
\left[\begin{array}{c|c|c|c|c}
B_k^{-1}A & B_k^{-1} & 0_{m\times r} & 0_{m\times r} & B_k^{-1}d \\
\hline
-C_B B_k^{-1}A + C & -C_B B_k^{-1} & -I_{r\times r} & I_{r\times r} & 0_{r\times 1} \\
\hline
C_B B_k^{-1}A - c & C_B B_k^{-1} & 0_{r\times r} & 0_{r\times r} & C_B B_k^{-1}d
\end{array}\right]
$$

The right hand corner of part (4) equal zero becouse $C_B^{(i-m)} B_k^{-1}d = C\bar{X}$ Comparing parts (4) and (5), it is clear that $y_{id} = -Z_j$ for $i =$ $m + 1, \ldots, m + r$ and J is an index of a nonbasic column. Since the values of $Z_j^{(i-m)}$ are the components of the rows of the objective functions for the point $\bar{X}$, then y_{ij}. can be found directly without recalculating the tableau. Thus the constructed problem can be initiated by replacing rows (5) of table (3) with a new criterial rows $(0_{1\times m} \; {-1}_{1\times r} \; 1_{1\times r} \; 0_{1\times r})$

Removing the artificial variables 0_{1xn} from the basis, we get:

Table (4)

$$
(6)\quad
\left[\begin{array}{ccccc}
B_k^{-1}A & B_k^{-1} & 0_{m\times r} & 0_{m\times r} & B_k^{-1}d \\
C_B B_k^{-1}A-c & C_B B_k^{-1} & I_{r\times r} & -I_{r\times r} & 0_{r\times 1} \\
1_{1\times r}(C_B B_k^{-1}A-c) & 1_{1\times r}(C_B B_k^{-1}) & 0_{1\times r} & I_{1\times r} & 0
\end{array}\right]
$$

artifical columns, can be omitted

The last row of table (4) is simply the sum of the r rows of the objectives. This row can be used to check optimality of problem (II) as well as the efficiency of the extreme point presented by the tableau. If there is a negative element in the last row, say the j - th, and all elements of the j - th column in rows (6) are negative, then for $\epsilon_j > 0$ Max V $>$ o and the corresponding extreme point $\overline{X} \notin E$.

Now, we present the technique used to enumerate the efficient extreme points of problem (I+) .

A Method for Generating all Efficient Extreme Points

Clearly, the set E is a subset of the set of all extreme points of $\underline{X}$. Since the latter set is finite, then consequently the number of efficient extreme points is finite too. Thus, it is possible to construct a method which can find such points. Here, we propose to give a computationally feasible procedure based on the standard simplex method which generates all efficient extreme points.

Let $H_q = (x \in R^n : x = (x_1, x_2, \ldots, x_n)$ is an extreme point of $\underline{X}$ and $X_q = 0$),

be the q hyperplane in the n - dimensional space R^n, q $\in (1, 2, \ldots n)$. We start the method by exploring all the efficient extreme points (if there is any) which lie on the facet $H_{r1} \cap \underline{X}$. After the registeration of all such points, we drop the hyperplane H_{rj} and continue searching for efficient points that may exist on the facet $H_{r2} \cap \underline{\overline{X}}_2$ of the convex polyhedron $\underline{X}_2$ and not on-

$H_{r1} \cap \underline{X}$. At the k-th stage, we would drop the hyperplanes $H_{r1}, H_{r2}, \ldots, H_{r_{k-1}}$, and search for efficient extreme points that may lie on the facet $H_{r_k} \cap \underline{X}_k$ and not on $H_{r_1} \cap \underline{X}$ or $H_{r_2} \cap \underline{X}_2, \ldots,$ or $H_{r_{k-1}} \cap \underline{X}$ k-1. The convex polynedron $\underline{X}_k$, where $1 \leq k \leq m$, is decribed by :

$$\sum_{j=1}^{n} a_{i_j} x_j = d_i, \text{ for all } i \neq r_1, r_2, \ldots, r_{k-1},$$

$$x_j \geq 0, \; j \neq r_1, r_2, \ldots r_{k-1}.$$

The process will come to an end in a finite number of steps since it must terminate when m of the hyperplanes are dropped, i.e, all required points will be found when at most hyperplanes are examined.

The previous idea can be applied by imptemening the following general rules:

Assume that we are in the k - th stage ; then:

1) Arbitrarly, we choose one of the nonbasic variables, say, X_{r_k} and we keep it in the nonbasic set throughout the current stage.

2) We examine all the extreme points of $\underline{X}_k$ and discard those which are not efficient.

3) We insert X_{r_k} into the basic set (if it is not possible, i. e, all components of the X_{r_k} - column are zeros, then all extreme points have been found, see (OMAR , 5)) and we hold it in the basic set till the end of the process.

4) We pick another nonbasic variable $X_{r_{k+1}}$, hold it in the nonbasic set, and then transfer to the next stage continuing from rule (2).

To carry out rule (2), we apply two methods, one for locating all extreme points of the convex polyhedra $\underline{X}_k$, $1 \leq k \leq m$, and the other for establishing the efficiency of each extreme point.

The general simplex tableau for the multisbjective problem (I) will be constructed as:

Table (5)

nonbasic / basic vars.	X_{N1}	X_{N2}	X_{Nn-m}	
X_{r_1}	Y_{11}	Y_{12}	$Y_{1(n-m)}$	$\hat{d}_1$
X_{r_2}	Y_{21}	Y_{22}	$Y_{2(n-m)}$	$\hat{d}_2$
⋮				⋮
$X_{r_{k-1}}$	Nopivots in k-th stage			⋮
X_{i_1}				⋮
X_{i_2}				⋮
⋮				
$X_{i_{m-k+1}}$	y_{m1}	y_{m2}	$Y_{m(n-m)}$	$\hat{d}_m$
f^1	Z_1^1	Z_2^1	Z_{n-m}^1	$\hat{f}^1$
f^2	Z_1^2	Z_2^2	Z_{n-m}^2	$\hat{f}^2$
⋮				
f^r	Z_1^r	Z_2^r	Z_{n-m}^r	$\hat{f}^r$
F	Z_1^{r+1}	Z_2^{r+1}	Z_{n-m}^{r+1}	$\hat{f}^{r+2}$ Composite Function

The last row represents the composite function $\sum_{i=1}^{r} C^i x$ which is used to check the efficiency of the current solution. $X_{N1}, X_{N2}, \ldots, X_{N_{n-m}}$ represent the nonbasic variables. Let us assume that all efficient extreme points of $\underline{X}$ that may lie on $H_{r_1} \cap \underline{X}$, $H_{r_2} \cap \underline{X}_2$, ..., and $H_{r_{k-1}} \cap \underline{X}_{k-1}$ have already been generated and thus the convex polyhedron $\underline{X}_k$ is left.

That is, the variables $X_{r_1}, \ldots, X_{r_{k-1}}$ are holded in the basic set Now, we are in the process of finding the efficient points of $\underline{X}_k$ which may lie on the hyperplane H_{r_k}, i.e, all efficient points of $\underline{X}_k$ for which $X_{r_k} = 0$. We call $\hat{X}$ a satisfactory point if it is an extreme point of $\underline{X}_k$ and not of $\underline{X}$. We call the m-k+1- tuples of the unordered integers ($i_1, i_2, \ldots, i_{m-k+1}$), $i_j \in (1, 2, \ldots, n)$, the indicator v of the extreme point X.

We Start the k-th stage by a satisfactory point $\overset{0}{X}$ of $\underline{X}_k$. we put the indicator V^0 of X^0 in a set R. By inspecting every nonbasic column, except the X_{rk} - column, of the simplex tableau corresponding to $\overset{0}{X}$, we can identify all neighbor indicators of $\overset{0}{V}$. In the k-th stage, we locate the extreme points of $\underline{X}$ lying on H_{r_k} but not on $H_{r_1}, H_{r_2}, \ldots$, or, $H_{r_{k-1}}$, thus , the elements lying in the basic rows X_{ri}, $i = 1, 2, \ldots, k-1$, must be holded as basic variables, i.e, the restrictions $X_{ri} > 0$ are ignored. We put in a set W all the new neighbor indicators $\Gamma(\overset{0}{V})$ of $\overset{0}{V}$. We choose an arbitrary element V^i from W and compute it, i.e, compute the satisfactory solution X^i corresponding to V^i. We check X^i for efficiency and out put -

it directly if it is efficient. Then we identify the new neighbors of V^i and put $R = V^i \cup \vartheta$ and $W = \Gamma(\vartheta) \cup \Gamma(V^i) - R$.

We pick another element from W and repeat the same process. At the s-th iteration we will have the two sets

$$R = \bigcup_{i=0}^{s} V^i \text{ and } W = \bigcup_{i=0}^{s} \Gamma(V^i) - R$$

The process will terminate when the set $W = \phi$ It holds that : if $w = \phi$ then R = the indicators of all extreme points of $\underline{X}_k$ (see , 4).

It is essential to consider the following cases:

(i) While constructing the neighboring indicators, if a tie occurs between some basic variables then all alternative basic variables must be considered in constructing the new neighbor indicators. Also, if some basic variables have zero values (a degenerate case), then each must be chosen in forming a neighbor idicator as soon as the corresponding element in the inspected column is nonzero.

(ii) If the elements of any of the currently investigated column are nonpoitive, then we leave it and move to the next column. Although, we do not obtain a new extreme point in the degenerate case, it is essential to create all different representations of the same degenerate solution because some may lead to new points in the subsequent steps.

It remains to present the techorique used to discard the extreme points which are not efficient.

We first check whether any of the objective functions, including the composite function, is at its maximum value at the current solution X^i. If at least one objective is uniquely maximized by X^i, then $X^i \notin E$. On the -

other hand, if $Z_j \leqslant 0$ for at least one nonbasic column and $\theta_j > 0$, this assures that $X^i \notin E$. However, if $Z_j \nleqslant 0$ for all nonbasic columns, then we have to establish the efficiency of the current solution. In this case we perform a number of simplex iterations on the criterial part, which is framed in table (5), of the simplex tableau. Each iteration is carried out with the largest positive coefficient to be the pivot element, of the nonbasic column having the most negative V in the (r+1)-th criterial row. If $\theta_j = 0$, we add the rows giving $\theta_j = 0$ to the criterial part and explore them after each iteration for any $y_{rj} > 0$. If there is $y_{rj} > 0$, then $\theta_j = 0$ and we perform the next iteration around y_{rj}. After a number of simplex iterations, one of the following two situations may occur:

(i) All coefficients of the (r+1) - th composite row are nonnegative, thus, in this case max $V=0$ and $X^i \in E$,

(ii) There is a negative element Z_j^{r+1} for which $Z_j \leqslant 0$ and $\theta_j > 0$, thus, in this case $X^i \notin E$.

Now we give a computational algorithm for the previous method..

Let S be an array of dimension mxu, where u is an upper bound of the number of extreme points of a convex polyhedron of dimension n-m-1. We divide S into two parts; the right part extends from the U-th column to S_1-th column, and the left part extends from the 1-st column to S_2-th column. We consider the most left nonbasic column of any simplex tableau as the X_{r_k}- column (any other column can be considered).

Step 1 . Start with an inital extreme point X^c of $\underline{X}$ and its indicator V^c. set k= 1.

Step 2. Store V^c in the u-th column of S. Set S_1=u-1 and S_2 = 1.

Step 3 Check the efficiency of X^c as follows:

(i) If $Z^i = (Z_1^i, Z_2^i, \ldots, Z_{n-m}^i) > 0$ for any $i \in (1, 2, \ldots, r+1)$, then $X^c \in E$ and hence out put X^c.

(ii) If $(Z_j^1, Z_j^i, \ldots, Z_j^r) \leq 0$ for any nonbasic $j \notin (1, 2, \ldots, n-m)$ and $\theta_j > 0$, then X^c is a nonefficient extreme point.

(iii) If conditions (i) and (ii) are not satisfied, then perform- a number of simplex ilerations on the criterial part of the simplex tableau as follows:

- Choose the column associated with the most negative element Z_j^{r+1} to be the pivot column.
- Perform the simplex operations with the largest positive element, say, Z_j^i, $i \in (1, 2, \ldots, r)$, to be the pivot element.
- If θ_j=0, add the corresponding rows to the criterial part, and perform the simplex iteration with y_{qj} as a pivot element if $y_{qj} > 0$. (The q - row is the one corresponding to the degenerate variable).
- After a number of iterations, either all components $Z_1^{r+1}, \ldots, Z_{n-m}^{r+1}$ will be nonnegative and so $X^c \notin E$, out-put X^c; or there is an element $Z_j^{r+1} < 0$ for which all coefficients $Z_j^1, Z_j^2, \ldots, Z_j^r$ are nonpositive and in this case $X^c \notin E$.

Step 4: Alternately, by moving systematically from the second column on the left to the right, inspect the nonbasic columns:

(i) If the last m-k+1 elements of the inspected column are non-positive, go to the next column.

(ii) If X^c is degenerate, create the neighboring indicator (s) of V^c corresponding to each degenerate variable.

(iii) If X^c is nondegenerate, determine the pivotal row of the inspected column and create the neighbor indicator of V^c.

For (ii) and (iii) if the created indicator (s) is neither in the left mor the right part of S, store the new indicator (s) in the S_2-th column (s) of s. Set $S_2 = S_2 + p$, where p is the number of the new indicators stored. Go to the next column.

(iv) If there is no more column to be examined, go to step 5.

Step 5: If $S_2 = 0$, go to step 7, otherwise, go to step 6.

Step 6: If at least one indicator was stored in S at step 4, pick the S_2-th indicator and compute it by carring out one simplex iteration. Move the S_2- th indicator to the S_1 - th column. Set $S_2 = S_2 - 1$ and $S_1 = S_1 - 1$. Go to step 3.

If no new indicators have been stored in step 4, choose from the left part of S that indicator, say, the h-th indicator, which has from V^c the least distance. Compute it by carring out a number of simplex iterations on the current tableau. Move the h-th indicator to the S_1 - th column and the S_2 - th indicator to the h - th column. Set $S_2 = S_2 - 1$ and $S_1 = S_1 - 1$. Go to step 3.

Step 7: If the last m - k+1 elements of the first column are zeros, terminate the process, other wise, go to step 8.

Step 8: Insert the 1st column into the basis. The pivot element is chosen from the entries of the last m-k+1 rows such that the new satisfactory point X^c must be an extreme point of $\underline{X}_{k-1}$, i.e, in determing the pivotal row, use the farmula:

$$\frac{X_{is}}{y_{is_1}} = \max_{j = 1, 2, \ldots, m-k+1} \left\{ \frac{X_{IJ}}{y_{ij_1}}, \; y_{ij}1 > 0 \right\}, \text{ if the}$$

last m-k+1 components are nonpositive, and the formula.

$$\frac{X_{is}}{y_{is_1}} = \min_{j = 1, 2, \ldots, m-k+1} \left\{ \frac{X_{ij}}{y_{ij_1}}, \; y_{ij}1 > 0 \right\}, \text{ if the}$$

last m-k+1 components are nonnegative.

Interchange the k-th row and the i_s - th row in the new tableau.

Step 9: Set k = k+1. If k > m, terminate the process. If d^c (The current values of the basic variables) $\geq$ 0, go to step 3 ; otherwise go to step 4.

An Example

Let us consider the following illustrative example:

To find all efficient extreme points of the linear multiobjective program:

$$\text{Max } f_1 = - x_1 - 2x_2 - X_3$$

$$f_2 = 2 X_1 + X_2 + X_3$$

subject to:

$$X_4 + \frac{1}{4}X_1 + \frac{3}{4}X_2 + \frac{1}{4}X_3 = 6$$

$$-\frac{1}{4}X_1 + \frac{1}{4}X_2 + \frac{3}{4}X_3 + X_5 = 14$$

$$X_j \geq 0 \text{ , } j = 1, 2, \ldots, 5.$$

In the following sequence of tableaux, asterisks denote the pivot elements which create new satisfactory solutions, and an arrow points to the hyperplane considered in the current stage. On the right hand side, we present the current contents of the set S. The composite function

$F = X_1 - X_2$ is denoted by Σ.

Stage 1

The initial simplex tableau is :

	X_1	X_2	X_3	d^c
X_4	$\frac{1}{4}$	$\frac{3}{4}$ *	$\frac{1}{4}$	6
X_5	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$ *	14
f_1	1	2	1	-6
f_2	-2	-1	-1	12
Σ	-1	1	0	6

S

$$\begin{bmatrix} 2 & 4 \ldots & 4 \\ 5 & 3 \ldots & 5 \end{bmatrix}$$

This extreme point $X^1 = (6, 0, 0, 0, 14)$ is efficient because the first objective function f_1 is at its unique maximum. So print x^1. Next we introduce X_3 into the basic set to get

	X_1	X_2	X_5	d^c
X_4	$\frac{1}{3}$	$2/3^*$	$-\frac{1}{3}$	4/3
X_3	$-\frac{1}{3}$	1/3	4/3	56/3
f_1	4/3	5/3	-4/3	-74/3
f_2	$\frac{-7}{3}$	-2/3	4/3	92/3
$\sum$	-1	1	0	6

S

$$\begin{bmatrix} 2 & 2 & \ldots & 4 & 4 \\ 5 & 3 & \ldots & 3 & 5 \end{bmatrix}$$

Since no objective function is at its optimal value, then we need to establish the efficiency of this point, so we proceed as follows:

f_1	$4/3^*$	5/3	-4/3
f_2	-7/3	-2/3	3/4
	-1	1	0

$\Longrightarrow$

f_1	4/3	5/4	-1
f_2	7/4	3	-1
	3/4	9/4	-1

Since $Z_5^i < 0$, i = 1,2 and $\theta_5 > 0$, then the extreme point $X^2 = (X_1^2, X_2^2, X_3^2, X_4^2, X_5^2) = (0, 0, 56/3, 4/3, 0)$ is not efficient we choose the neighbor (2,3) and compute it, So:

	X_1	X_4	X_5	
X_2	1/2	3/2	$-\frac{1}{2}$	2
X_3	$-\frac{1}{2}$	$-\frac{1}{2}$	3/2	18
f_1	$\frac{1}{2}$	-5/2	$-\frac{1}{2}$	-28
f_2	-2	1	-1	32
	-3/2	-3/2	$\frac{1}{2}$	4

S

$$\begin{bmatrix} 2 & \ldots & 2 & 4 & 4 \\ 5 & \ldots & 3 & 3 & 5 \end{bmatrix}$$

We check the efficiency of this point:

f_1	$\frac{1}{2}$ *	-5/2	$-\frac{1}{2}$
f_2	-2	1	1
	-3/2	-3/2	$\frac{1}{2}$

$\Longrightarrow$

f_1	2	-5	-1
f_2	4	-9	-1
	3	-9	-1

Because of the second column and $\theta_4 > 0$, then $X^3 \notin E$.

We continue by computing the indicator (2,5) :

	X_1	X_4	X_3	
X_2	$\frac{1}{2}$ *	4/3	$\frac{1}{3}$	8
X_5	$\pm\frac{1}{3}$	$-\frac{1}{3}$	2/3	12
f_1	1/3	-8/3	1/3	-22
f_2	5/3	4/3	-2/3	20
	$-\frac{4}{3}$	$-\frac{4}{3}$	$-\frac{1}{3}$	-2

$$S = \begin{bmatrix} \cdots\cdots & 2 & 2 & 4 & 4 \\ \cdots\cdots & 5 & 3 & 3 & 5 \end{bmatrix}$$

f_1	1/3*	- 8/3	$\frac{1}{3}$
f_2	-5/3	4/3	-2/3
	-4/3	-4/3	$-\frac{1}{3}$

$\Longrightarrow$

f_1	3	-8	1
f_2	5	-36/3	1
	4	-12	1

$X_4 \notin E$, because $\theta_4 > 0$ and $Z_4^i < 0$. The left part of S is empty, thus stage 1 is ended. We start stage 2 by calculating the indicator (1, 5) The variable X_1 must be holded in the basic set and the hyperplane $X_4 = 0$ is invistigated in this stage.

Stage 2:

	X_4	X_2	X_3	
X_1	4	3	1	24
X_5	1	1^*	1^*	20
f_1	-4	-1	0	-30
f_2	8	5	1	60
$\sum$	4	4	1	30

$$S \quad \begin{bmatrix} 1 & 1 \dots & 1 \\ 2 & 3 \dots & 5 \end{bmatrix}$$

Print $X^5 = (24, 0, 0, 0, 20)$, since f_2 is at its unique maximum value. We compute (1, 3), then:

	X_4	X_2	X_5	
X_1	3	2	-1	4
X_3	1	1	1	20
f_1	-4	-1	0	-30
f_2	7	4	-1	40
	3	3	-1	10

$$S \quad \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \end{bmatrix}$$

Since $Z_5^i \leq 0$ and $\theta_5 > 0$, then the point is not efficient. We compute (1, 2), then:

	X_4	X_3	X_5	
X_1	1	-2	-3	-36
X_2	1	1	1	20
f_1	-3	1	1	-10
f_2	3	-4	-5	-40
	0	-3	-4	-50

S

$$\begin{bmatrix} \cdots & 1 & 1 & 1 \\ \cdots & 2 & 3 & 5 \end{bmatrix}$$

This point is not efficient sice it is not an extreme point of $\underline{X}$ the left part of S is empty, so stage 2 is finished. The number of hyper plane explared is two, thus all extreme points of $\underline{X}$ have been bound.

Computational Experience

The algorithm has been coded in FORTRAN V and used to run a number of test problems on the INTERBATA 7/32 computer, Computing Department, Institute of National Planning! The problems were of different sizes. The largest problem Solved was of 13 constraints, 23 variables, and 8 objective functions. It took 5/5 Simplex 8 teps to find 104 efficient extreme points. The computing time was 74 minutes. Of course, the computing time should be only of relative value since it could be changed if the program is executed on other models of computers. Some of the test problems have been terminated i.e, all efficient points are found, before all the hyperplanes are examined. A testing degenerate problem having 5 constraints, 10 variables, and H objective functions gave to efficient points, one of them had been printed 3 times because of its degeneracy (see appendix).

I mention some of the difficulties which I have faced. First, since the indicators of the extreme points are to be saved, so the determining factor of the problem to be solved is the computer stovage capacity. The space needed for the set S is of dimension m X u, where

$$U = \binom{n + m - \left[\frac{n+2}{2}\right]}{m} + \binom{n + m - \left[\frac{n+3}{2}\right]}{m}$$

for a convex polyhedron in the n-dimen- Sional space. The formula used for the upper bound u is given in (   ). This formula is not yet proved for all Values of mand n but it gives considerably better results than the other known formula. Since the proposed method investigates one of the polyedra H_{r_i} n $\underline{X}_i$ at a time, that is, the space dimension is reduced by one, then the storage area of S could be reduced by:

$$m \; X \; \frac{2m}{n-1} \binom{m + \frac{n-3}{2}}{m} \qquad \text{locations.}$$

However, although some of the testing problems theoritically required a storage area which exceeds the available core capacity, we could run most of them without invoking auxiliary storage. This could be done by using the value of U which fits the data into the main core and including a device into the program which gives a warning message in case more storage locations are required.

The second difficulty is introduced by the fact that many extreme points, in case of lengthy problems, must be calculated so that round-off errors may accumulate to the point where they obscure the actual results. This problem may be resolved it double precision arithmetic is used to improve the accuracy.

In the appendix the results of ~~two~~ some problems are presented.

APPENDIX

```
          OBJECTIVE FUNCTION    1 AT ITS MAX VALUE
  X  2       6.00000000
  X  3      14.00000000
 VALUES OF OBJ.  FUN.
   -6.00000000
   12.00000000
          OBJECTIVE FUNCTION    2 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    3 AT ITS MAX VALUE
  X  4      23.99998474
  X  3      19.99998474
 VALUES OF OBJ.  FUN.
  -29.99998474
   59.99998474
   NO. OF EFFICIENT PT. =  2NO. OF EXT. PTS. =  7NO. OF HYPERPLAN INVI

END OF PROGRAM
```

```
X  3        2.00000000
X  4        1.00000000
X  5        0.00000000
X  6        3.00000000
X  7        1.00000000
 VALUES OF OBJ.  FUN.
     0.00000000
     0.00000000
     0.00000000
     0.00000000
X  3        1.00000000
X  4        0.50000000
X  5        0.99999994
X  2        0.99999994
X  7        0.00000006
 VALUES OF OBJ.  FUN.
     0.99999994
     0.50000000
    -0.99999994
    -0.50000000
X  3        2.00000000
X  4        1.00000000
X  6        3.00000000
X  2        0.00000000
X  7        1.00000000
 VALUES OF OBJ.  FUN.
     0.00000000
    -0.00000006
     0.00000000
     0.00000006
X  1        0.50000000
X  4        1.00000000
X  6        0.25000000
X  2        1.00000000
X  3        0.50000000
 VALUES OF OBJ.  FUN.
     2.00000000
     0.74999994
    -1.00000000
    -0.99999994
X  1        0.00000000
X  4        0.49999976
X  5        1.00000000
X  2        0.99999994
X  3        1.00000000
 VALUES OF OBJ.  FUN.
     1.00000000
     0.50000000
    -0.99999994
    -0.49999994
X  1        0.00000286
X  7        0.99999976
X  4        1.00000381
X  6        3.00000000
X  3        1.99999905
 VALUES OF OBJ.  FUN.
    -0.00000095
     0.00000238
     0.00000006
     0.00000000
    NO. OF EFFICIENT PT. =   6NO. OF EXT. PTS. = 11NO. OF HYPERPLAN INVISTIGATED=
```

```
          OBJECTIVE FUNCTION   2 AT ITS MAX VALUE
 X  5      2.00000000
 X  6      1.50000000
 X  7      2.50000000
 X  8      0.49999988
 X  9      4.00000000
 X 10      1.00000000
 X 11      0.50000012
 X 12      2.99999905
 VALUES OF OBJ. FUN.
    -3.49999905
     2.50000000
    -1.00000000
    -0.00000006
     0.50000000
          OBJECTIVE FUNCTION   3 AT ITS MAX VALUE
          OBJECTIVE FUNCTION   4 AT ITS MAX VALUE
          OBJECTIVE FUNCTION   5 AT ITS MAX VALUE
          OBJECTIVE FUNCTION   6 AT ITS MAX VALUE
 X  1      1.00000000
 X  6      0.00000000
 X  7      4.00000000
 X  8      0.00000000
 X  9      4.00000000
 X 10      1.00000000
 X 11      0.50000000
 X 12      2.99999905
 VALUES OF OBJ. FUN.
     3.00000000
     2.00000000
     5.00000000
     4.00000000
     1.00000000
 X  1      1.00000000
 X  4      0.00000000
 X  7      4.00000000
 X  8      0.00000000
 X  9      4.00000000
 X 10      1.00000000
 X 11      0.50000000
 X 12      2.99999905
 VALUES OF OBJ. FUN.
     3.00000000
     2.00000000
     5.00000000
     4.00000000
     1.00000000
          OBJECTIVE FUNCTION   4 AT ITS MAX VALUE
 X  1      1.00000000
 X  3      0.00000000
 X  7      4.00000000
 X  8      0.00000000
 X  9      4.00000000
 X 10      1.00000000
 X 11      0.50000000
 X 12      2.99999905
 VALUES OF OBJ. FUN.
     3.00000000
     2.00000000
```

```
        [illegible]
        4.00000000
        1.00000000
            OBJECTIVE FUNCTION    4 AT ITS MAX VALUE
  X  1        1.00000000
  X  2        0.00000000
  X  7        4.00000000
  X  8        0.00000000
  X  9        4.00000000
  X 10        1.00000000
  X 11        0.50000000
  X 12        2.99999905
   VALUES OF OBJ. FUN.
        3.00000000
        2.00000000
        5.00000000
        4.00000000
        1.00000000
  X  1        1.00000000
  X  3        0.00000000
  X  7        4.00000000
  X  4        0.00000000
  X  9        4.00000000
  X 10        1.00000000
  X 11        0.50000000
  X 12        2.99999905
   VALUES OF OBJ. FUN.
        3.00000000
        2.00000000
        5.00000000
        4.00000000
        1.00000000
  X  1        1.00000000
  X  2        0.00000000
  X  7        4.00000000
  X  4        0.00000000
  X  9        4.00000000
  X 10        1.00000000
  X 11        0.50000000
  X 12        2.99999905
   VALUES OF OBJ. FUN.
        3.00000000
        2.00000000
        5.00000000
        4.00000000
        1.00000000
            OBJECTIVE FUNCTION    1 AT ITS MAX VALUE
  X  1        1.00000000
  X  6        0.00000000
  X  7        4.00000000
  X  3        0.00000000
  X  9        4.00000000
  X 10        1.00000000
  X 11        0.50000000
  X 12        2.99999905
   VALUES OF OBJ. FUN.
        3.00000000
        2.00000000
        5.00000000
        4.00000000
        1.00000000
            OBJECTIVE FUNCTION    3 AT ITS MAX VALUE
            OBJECTIVE FUNCTION    5 AT ITS MAX VALUE
            OBJECTIVE FUNCTION    2 AT ITS MAX VALUE
  X  1        1.00000000
```

```
X  7       4.00000000
X  3       0.00000000
X  9       4.00000000
X 10       1.00000000
X 11       0.50000000
X 12       2.99999905
 VALUES OF OBJ. FUN.
    3.0000000
    2.0000000
    5.0000000
    4.0000000
    1.0000000
X  1       1.00000000
X  5       0.00000000
X  7       4.00000000
X  3       0.00000000
X  9       4.00000000
X 10       1.00000000
X 11       0.50000000
X 12       2.99999905
 VALUES OF OBJ. FUN.
    3.0000000
    2.0000000
    5.0000000
    4.0000000
    1.0000000
          OBJECTIVE FUNCTION    1 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    3 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    4 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    5 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    6 AT ITS MAX VALUE
X  1       1.00000000
X  5       0.00000000
X  7       4.00000000
X  3       0.00000000
X  9       4.00000000
X 10       1.00000000
X 11       0.50000000
X 12       2.99999905
 VALUES OF OBJ. FUN.
    3.0000000
    2.0000000
    5.0000000
    [illegible].0000000
    1.0000000
          OBJECTIVE FUNCTION    1 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    3 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    4 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    5 AT ITS MAX VALUE
          OBJECTIVE FUNCTION    6 AT ITS MAX VALUE
   1       1.00000000
   5       0.00000000
   3       0.00000000
   7       4.00000000
   9       4.00000000
  10       1.00000000
  11       0.50000000
  12       2.99999905
VALUES OF OBJ. FUN.
    3.0000000
    2.0000000
    5.0000000
    4.0000000
    1.0000000
```

```
END OF PROGRAM




 X  4      10.00000000
 X  3       4.00000000
 X  6      19.00000000
 X  7       5.00000000
 X  8       6.00000000
  VALUES OF OBJ. FUN.
     0.00000000
     4.00000000
     4.00000000
          OBJECTIVE FUNCTION   2 AT ITS MAX VALUE
          OBJECTIVE FUNCTION   3 AT ITS MAX VALUE
 X  4       5.00000000
 X  3       3.00000000
 X  6      15.00000000
 X  2       2.00000000
 X  8       3.00000000
  VALUES OF OBJ. FUN.
    -2.00000000
     5.00000000
     5.00000000
          OBJECTIVE FUNCTION   1 AT ITS MAX VALUE
          OBJECTIVE FUNCTION   4 AT ITS MAX VALUE
 X  1       2.66666698
 X  6       3.00000286
 X  7       4.99999905
 X  3       2.00000095
 X  8       5.33333206
  VALUES OF OBJ. FUN.
     8.00000477
     2.66666603
    -0.66666657
 X  1       1.33333492
 X  6       7.00000286
 X  2       1.99999809
 X  3       2.00000095
 X  8       2.66666698
  VALUES OF OBJ. FUN.
     2.00001431
     4.33333111
     2.66666317
    NO.OF EFFICIENT PT =  4NO.OF EXT PTS = 15NO.OF HYPERPLAN INVISTIGATED=  5

END OF PROGRAM
```

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