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Optimal And Approximate Ordering In Solving Investory Problems

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Introduction

Since the development of the original lot- size inventory model in 1915. many researches was done analysing mathematical models for describing ordering policies for one or more products Invariably, it was assumed implicitly that once units enter in to inventory, they (live) for ever or else they expire after only a single planning period.

In most industrial environments, planning over the short run is essentially inaffected by problems of obsoles cence, and the inifite-lifetime assumptions is not unreasonable.

How ever, there is a significant class of problems for which the perishable nature of the inventory cannot be ignored in thedeve-loping optimal ordering policies.

In the private sector the food industry is the most solient example of one concerned almost exclusively with inventory problems of a perishable nature. Inventory managment is required at virtually every level of the food chain. Determining optimal planting policies, stocking policies at inter mediate ware house locations. and inventory policies in the retail market place are some exampels Although many canned or frozen products may be considered as having an essentially infinite lifetime, fresh produce, meat, etc are hig-highly perishable in nature.

There are a number of significant inventory managment problems arising in many field such as the field of health adimistnation that deal with perishable inventories- Avery interesting case is blood banking.

The type of model we will deal with is as follows:

At the start of each period an order is placed for any number of fresh units, which are delivered immediately. During the period a random demand D(with known probability distribution F and density f is either satisfied by avialable stock or back logged.

In practice it is for more realistic to allow the ordering cost to be composed of both fixed and proportional components; that is, it is of the form K+CY for Y > 0, and 0 for y=o

The purpose of other dealing of such model is to consider the effect of the fixed charge, K, on the nature of the optimal ordering policy funcation.

Also we deal with the problem of determining both optimal and opproximate ordering policies for a fixed- life perisable commodity when there is a fixed charge or set- up cost for placing an order. our motivation is to construct on (s , S) type approximation.

1- Analysis of the problem,

For the sake of clarity, we state our assumption explicitly

1 - All orders are placed at the start of the period and received instantly(that is, Zero lead time)

2 - Demand is successive periods are independent identically distributed random variables with common distribution F and density f. in addition, $f(o) > o$ for $x > o$

3 - All stock arrives new.

4 - Inventory is depleted according to a (First in,First out) policy

5 - The following are the relevant costs:

a - ordering of C per unit ordered

b - set up of K per order

c - out dating of Q per unit that perishes.

d - shortage charged against the number of units that go short.

6 - if after m periods ($m \geq 2$) a unit has not been depleted by demand, it must be discarded at a cost given in $\delta(c)$

7 - All excess demand is back logged

We define X_i = amount of stock in inventory that is scheduled to out date in exuctily i period(or equivalently that was aquired m-i period ago). The state variable will be the vector quantity.

$$X = (x_{m-1},\ x_{m-2}, \ldots\ldots, x_1)$$

we will let y be the quantity of new product. The natation

$$x(K) = (x_K, \ldots, x_1) \qquad \text{for } 1 \leq K \leq m$$

will be also used, where η will be understood to be y in this context . Normally, the solution to a one -period model would not take into account the effects of out dating since units do not expire for m periods However, it is possible to include in an explicit way the effect of out dating by noting that the number of units of the current order y that will out date after m period is a random variable that depends upon both the vactor x and the realiziations of the demand over the next m periods. This is du to the fact that since we are using a (first input - first out put) depletion policy, all of y will necessarily have been used to satisfy demand before any futher order. It is demonstrated in (3) and (4) that the expected number of units of the current order y sheduled to out date after m periods of demands is given by

$$\int_0^y G_m (u , x)\, du \text{ ; where}$$

$$G_n\, x(x(n)) = \int^{x_n} G_{n-1} (\; + x_{n-1} , x_{(n-2)}\; f (x_n - \xi)\, d\xi)$$

for $2 \leq n \leq m$

and $G_1 (x) = F(x_1)$

This is generally an extremely tedious compution which would have to be approached using numerical methods.

Formal analysis of the one period model will require a number of results concerning the G_n functions their derivatives we will use the following notation;

If h is a function of an n dimensional vactor U, then $h^{i}(u)$ represents the partial derivative of h with respect to its i th

te argument note that if

$$\mu = (\mu_n, \ldots\ldots, \mu_1) \text{ then}$$
$$h^{(i)} = \frac{\partial h(u)}{\partial x_{n-i+1}}$$

Also we let the functions $H_n(x(n))$ which can be computed by the recursion

$$H_n(x(n)) = \int_0^{x_n} H_{n-1}(\gamma + x_{n-1}, x(n-2)) f(x_n - \gamma)\, du - H_{n-1}(x(n-1)) F(x_n)$$
$$\text{for } 2 \leqslant n \leqslant m$$

and $H_1(x_1) = F(x_1)$

by differentioting both sides of the about equation with respect to x_n and integrating by ports, one obtain

$$H_n^{(1)}(x(n)) = \int_0^{x_n} H_{n-1}^{(1)}(\gamma + x_{n-1}, x(n-2)) \cdot f(x_n - \gamma)\, du$$

Since $H_1^{(1)}(x_1) = f(x_1) \geqslant 0$, it follows by induction that $H_1^{(1)}(x(n)) \geqslant 0$ or equivalently that $H_n(x(n))$ is non - decreasing in the variable x_n. The following results indicutes the relationship between the functions H_n and G_n and can be established by an induction arqument.

Taking the following expression

(1) $$\int_0^y G_m^{(i)}(\mu, x)\, du = G_m(y,x) - \sum_{j=1}^{i-1} G_{m-j}\,(x(m-j)) H_j(y, \bar{x}(m-j))$$

for $2 \leq i \leq m$ where

$\bar{x}(m-j) = (x_{m-1}, x_{m-2}, \ldots\ldots, x_{m-j+1})$

The left hand side of (1) represents the rate of change of hte expected out dating with respect to x_{m-i+1}, which is on hand stock (i -1) period old.

The first term on the right - hand side is the rate of change of the expecled out dating with respect to y, which is new stock , hence one interperetation of exspression(1) is that the expected out dating increases more rapidly in y than in each component of x. Note that expression 1 also implies that for $y > 0$

$$\int_0^y \left[G_m^{(i)}(\mu, X) - G_m^{(i-1)}(\mu, x)\right] du$$
$$= G_{m-1}\,(x\,(m-i)) H_i\,(y, \bar{x}\,(m-i)) > 0$$

for $3 \leq i \leq m$

which mean that the expected out dating increases more rapidly with Z_{m-i+1} than X_{m-i} in other words the expected number of units of the current order y to out date in m period is more sensitive to change in the current in ventory of never stock than the older stock. This property must be reflected in the optimal ordering policy function as well

2 - A one - period Model with a set - up cost

The purpose of looking at a single - period model is to determine the effect of out dating on the structure of the optimale policy when a fixed charge $K > 0$ is present. By substituting on expected out dating cost to be incurred m periods, rather than merely computing the expected cost in curred in the present period, the first - period the first- period model will take explicit account of the process dynamics. In approaching the $K = 0$ case in this has hion, H. ScARf (S) demonstrates that the optimal policy for the multt period model possesses all the properties of the optimal policy for the multi dynamic period model. It seems reasonable to conjecture that this holds when $K > 0$ as well computational experience bears out this conjecture. The structural results obtained are then used to motivate a suitable from for a simple approximation. Now let

$X = \sum_{i=1}^{m-1} x_i$, which has the usual interpretation as the total inventory an hand before ordering. A ssuming that the cost of out dating of y is paid for when the order is placed, the expected present cost of ordering, set-up holding, shortage and outstanding is

$$cy+k\delta(y) + \int_0^{x+y} h(x+y-t)\, f(t)dt + \int_{x+y}^{\infty} p(t-x-y)f(t)dt + \theta\int_0^{y} G_m(u,x)du$$

when the cost of outstanding is paid for in the period of the out dating actually occurs, one must replace θ by $x^{m-1}\theta$ in the final term since the outdating charge then becomes a future cost. chazan (6) discussed this point in details.

Now let

$$L(z) = \int_0^z h(z-t)\, f(t)\, dt + \int_z^y p(t-z)\, f(t)\, dt$$

and

$$B(x,y) = \begin{cases} cy + L(x+y) + \theta \int_0^y G_m(u, x)\, du & \text{for } y \geqslant 0 \\ L(x+y) & \text{her } y < 0 \end{cases}$$

L(z) has the usual interpretation as the expected one- period cost of holding and shortage when Z is the inventory on hand after ordering.

When $y \geqslant 0$, B(x,y) is merely the expected one period costs of ordering Y excluding the cost of set-up. It is conuenient to extend B(x,y) as a function of y to the entire whole line as indicated above even though disposal will not be allowed as a policy option.

By differentiating, it is easy to see that B(x , y) is convex in y for all fixed vectory x. We will denote by y (x) the value of y that minimzes B (x,y). A necessary and sufficient condition that $y(x) > 0$ is that $x < \bar{x}$, where $\bar{x}$ solues $c + L'(\bar{x}) = 0$.

jf $x < \bar{x}$ and its is aptimal to place an order, then it is qstimal to order y(x) and incur an expected cost of

$$K + B(x, y(x))$$

If no order is placed, then an expected cost $B(x,o)$ is incurred.

Define $S (x)$ to solve: $B(x,y(x) + K, s(x) \leqslant y(x)$.

from figure (1) it is clear that $s(x) < y(x)$ and the lollowing eqnivalence hold because of convexity of $B (x,y)$ in y

$$s(x) > o \leftrightarrow B(x,o) > B(x,y(x))+ K$$

$$s\ x \leqslant o \leftrightarrow B(x,o) \leqslant B(x,y(x))+ K$$

Hence it is optimal to place an order for $y(x)$ of new product if and only if $s(x) > o$. The funcation $y(x)$ determines the optimal order quantity, while $s(x)$ determines the ordering reqion . The optimal ordering function $y(x)$ possesses the following property:

$$o > y^{m-1}(x) \geqslant y^{m-2} \geqslant \cdots \geqslant y^{1}(x) \geqslant -1$$

This may be seen from the result of (S)

This may be interpreted as follows:

If one increases the stock on hand before ordering by one unit (of any age), then the optimal ordering quantity will decrease but by less than one full unit. In addition, the optimal ordering policy function is more sensitive to changes in newer inventory than older inventory: if one increase x_i, then $y(x)$ decreases more than if one increases x_j by the same amount whenever $i > j$.

We will show that the function $s(x)$ possesses essentially the same structure as y(x). In order to do this we need the hollowing:(s)

$1 - B_i^{(i,m)}(x,u) > o$ for $1 \leqslant i \leqslant m$ and for all x and $\mu > o$

$2 - B_i(x,u) = B^{(m)}(x,u) - c\text{-}o \sum_{j=1}^{i} G_{m-j}(X(m-j))\, H_j(\mu, \bar{x}(m-j))$

$3 - o > s^{(m-1)}(x) \geqslant s^{(m-2)}(x) \geqslant \ldots\ldots\ldots \geqslant s^{(1)}(x) \geqslant 1$

Depending an the smoothness of B (x,y), the function s(x) and y(x) may be essentially parallel surfaces.

For m = 2 case, the situation is as in fig2.

The functions y(x) and s(x) are pictured as the projections outs the (x,y) plane .

it is interseting to compare these results with $K = o$ case . For $K=0$ the decision of whether to place an order depenent on x through the sum x, In that case , if the total inventory an hand entering any period was less than $\bar{x}$ then it was optimal to place an order. However when $k > o$, the ordering region becomes much more complex. The region boundary is given by the locus of points ($x \in R^{m-1}$: $s(x) = o$), which will be a continuously differentiable hyber- surface in Euclidan (m-1) space.

when m=2 the state vector x and x are identical. The set($x \in R^1$: $s(x)=o$) consists of exuctly one point which means that for the m=2 case only there is a constant $s^* < \bar{x}$, such that it is optimal to place an order if and only if $x < s^*$.

Only for $m = 2$ close the optimal ordering region have the same form as in the non-perishable case- it seams likely that similar hold for the multiperiod dynamic problem.

In the following part we will show how to construct two simple approximations and compare thier performance to the optimal policy.

3- An (s , S) approximation

The first problem encounteved when coustructing on approximation is the specification of the form of the approximate policy . Since it is well known that an (s,S) policy is optimal for the corres ponding non- perihable problem this would seem like a reasonable form for an approximate policy for perishable problem. There is additional evidence for its suitability. From (7) and (8) , a ciritcal number or S policy (that is one that requires ordering to a fixed level each period) provides an excellent approximation to the optimal order quantity for $K=0$. That would lend to imply that the function y(x) can be closely approximated by S-X. Since s(x) and y(x) posses essentiaty the same structure, s(x) can also be closely approximated by the term s-x. The ordering region, nomely $(x: s(x) > 0)$, becomes the region $x \leq s$.

Hence one obtions the (s,S) approximation:

if $x \leq s$ order to S; otherweise, do not order. Having determined a suitable form for approximeted at best we would like to obtain the optimal(s,S) policy. However, this appears to be an extermely difficult problem. For simpler case of $K=0$ cohn(10) could obtain an explicit expression for the stationary distribution of the stock level only for the $m=2$ case. This computation would be substantially more difficult for $K > 0$

In fact, determining the optimal (s, S) policy might be more difficult than determing the optimal policy it self.

Another possibility would be to determine an approximate (s,S)policy Since we would be approxmating in two directions(funding a suboptimal policy from a class of suboptimal policies) , It would be important to compare the performance of the approximation with that of the optimal policy .

Let $Z = x + Y$ represent the total quantity of inventory on hand after ordering (z may be thpugh of as an order up- to point). we will require

1 - The expected one - period costs of ordering, holding, shortage, and outdating to be approximated by a separable function of the variables x and z

(Recoll $x = \sum_{t=1}^{m-1} x_i$)

2 - The transter function, $t(x, y, D) = t_{m-1}(x, y; D); \ldots\ldots$

$t_1(x; y; D)$,

the vector of starting in ventories one period hence , when x is the current vector of inventories, y is the order quantity, and D is the demand in the period.

Formally, for $1 \leq j \leq m-2$

$$t_j(x, y\, D) \begin{cases} - x_{j+1} & \text{if } 0 \leq D \leq \sum_{j=1}^{j} x_j \\ -y + \sum_{i=1}^{j+1} x_i - D & \text{if } \sum_{i=1}^{j} x_i < D \leq \sum_{i=1}^{j+1} x_i \\ - 0 \text{ otherwise} & \end{cases}$$

and $$t_{m-1}(x ; y ; D) = \begin{cases} y & \text{if } D < x \\ y + x - D & \text{if } D > X \end{cases}$$

This form assumes full backlogging of demand.

The rationale for the approximate trans fer functions that we will construct is based on the fact that both the starting inventories in each period and the number of units out dating each period are random variables whose distributions converge when following a stationary(s,S) policy. (cohen(2) has developed rigorous orguments to prove this for the case s = S /.

We expect that similar results should hold for K $>$ o .)

For convenience, we will hence forth use the notation x_n, n $\geq$ to pepresent the total inventory on hand at the start of period n. when following a stationary policy x_n will form a sequence of random variables that converge in distribution . Since our interest is in constructing a stationary approximation, we treat x_n and x_{n+1} as approximately identically distributed.

let $V(z) - \hat{V}(x)$ represents the opproximate expected one - period costs of ordering , holding, shortage and out dating, and $t(z,D)$ represent the approximate transfer function (that is , $B(x,y \approx V(x) - \hat{V}(x)$ and $\sum_{i=1}^{m-1} t_i (x , y , D) \approx t(z,D)$

optimal policies for the opproximate model will sotisfy the functional equations :

$$G_n(x) = \min_{Z \geq X} \left\{ k \delta(z-x) + v(Z) - \hat{V}(x) + \alpha \int_0^\infty C_{n+1}(t(Z,u)) \right.$$

for $1 \leq n \leq N$ and $C_{N+1}(.) = 0$

Note that periods are numbered forward and N is the length of the planning horizon. The acutnal for v(z) , $\hat{v}(x)$ and t(z,u) will depend upon the particular approximate forms used for expected out - dating. In general, it is difficult to prove that an (s,S) policy is optimal for a model of this form.

Scorf,s (10) original approach would require proving inductively that $V(z) - \hat{v}(x) + \alpha \int_0^\infty C_{n+1} \; (t\,(z,u,))f(u)\,du$ -k- konvex in the variable z.

For one of our models we can show that $c_N(x)$ is not convex for all values of $k = 0$.

However, by using a clever device developed by veinott (16), we can convert this model to an equivalent one having only holding, shortage, and set up costs. To do so we will need the additional assumption that there is a return of $\hat{v}\,(x_{N+1})$ at the end of the horizon. The physical signifi conce of this assumption will be discussed rdative to the two forms of $\hat{v}(x)$ considered .

The total (approximate) expected discounted cost for N periods may be written as

$$E\left\{\sum_{n-1}^{N} \alpha^{n-1} \left(KS\,(z_n - x_n) + V(z_n) - \hat{V}(x_n) - \alpha^n \hat{v}(x_{N+1})\right)\right\}$$

$$= E\left\{\sum_{n-1}^{N} \alpha^{n-1}\left(K\delta(z_n - x_n) + V(z_n) - \alpha\hat{V}(x_n)\left(t\left(z_n, D_n\right)\right)\right)\right\} - \hat{v}(x_1)$$

which results from using the relationship

$$x_{n+1} = t\,(z_n, D_n)$$

and shifting the term $\hat{v}\,(x_n)$ back one period the starting inventory, x_1, is a fioxed constout.

Define

$$w(z) = v(z) - \alpha E(\hat{v}(t(z,D))) \;.$$

Then the total expeced discounted cost for the N periods may be wrillen

$$E\left\{\sum_{n=1}^{N} \alpha^{n-1}\left[k\,\delta(z_n - x_n) + W(z_n)\right.\right. - \hat{V}(x_1)$$

The final term beignored as it is aconstant that dose not affect the computation of the optimal policy.

The optimal salution of the approximate model will now satisfy the functional equations

$$C_n(x) = \min_{z \geq x}\left\{K\,\delta\,(z-x) + W(z) + \alpha\int_0^{\infty} c_{n+1}(t(z,u)\,f(u),\,du\right.$$

We will consider two different forms for the approximate one - period cost and transfer function.

In (9) it is demonstrated that

$$G_m(\mu.x) \leqslant F^{m*}(\mu+x)$$

where F^{m*} is m hold convolution of the one - period demand distribution , F. It follows that the expected outdatingmmay be approximated by $H(z)-H(x)$ where

$$H(t) = \begin{cases} \int_0^t f^{m*}(u)\,du & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0 \end{cases}$$

from this bound it follows that

$$V(z) = cz + L(z) + \theta H(z) \text{ and}$$

$$\hat{V}(x) = cx - \theta H(x) \quad .$$

The approximate transfer functions are obtained by using the identity:

$$x_{n+1} = z_{n+1} - D_n$$

number of units perishing in period, n. In this case we have

$$x_{n+1} \approx z_n - D_n + H(z_n) \approx z_n - H(z_n) + H(x_{n+1})$$

Since x_{n+1} and x_n are respecticely opproximated distributed for longe n, This sugges is that the approximate tranfer function satisfy the functional equation

$$t(z_n; D_n) = z_n - D_n - H(z_n) + H(t(z_n; D_n))$$

This appears to be a dificult equation to solve, but since $H(z_n)$ is likely to be small compared to z_n,

$$H(t(z_n - D_n) \approx H(z_n - D_n$$

and we obtain the approximation

$$(z_n; D_n) = z_n - D_n - H(z_n) + H(z_n - D_n)$$

It now follows that, ignoring coustout

$$W(z) = ((1-\alpha) z + L(z) + (\theta + \alpha c) H(z) - \alpha (\theta + c) \int_0^x H(z-u) f(u)\, du,$$

where the approxmation

$$H(z-D-H(z)+H(t(z,D))) \approx H(z-D)$$

was again used in the final term. Note that by an integration by parts

$$\int_0^z H(z-u) f(u)\, du = \int_0^z F^{(m+1)*} (u) du.$$

The solvage value as sumption for this model (that there is a return of $\hat{V}(x_{n+1})$ at the end of the horizon) corresponds to a return of cx_{n+1} + $QH(x_{n+1})$ at the end of the herizon if $x_{n+1} > c$ and an additional cost of $-c\, x_{n+1}$ if $x_{n+1} < o$ since we have implictely assumed that the out dating cost is paid for when the order arrives, the term $QH(x_{n+1})$ may be considered to be a return an the out dating charge on stock that never actually out dates under the assumption that one order to z each period. Chazan and Gol (1) show that the ~~expected~~ outdating per period is bounded below by

$$Z/m - a(z)$$

and above by $z/m - b(z)$, where

$$a(z) = \int_0^{z/m} m\,x\, f^{m*}(mx)\, dx + (z/m)(1-F^{m*}(z))$$

and

$$b(z) = \int_0^{z/m} xf(x)\, dx + (z/m)\,(1 - F(z/m))$$

it is easy to see that a (x) is the expected average demand over m period turncuted at z /m and b(z) is the expected demand in one period truncated at z/m. Note that by an integration by parts one olstains.

$$a(z) = (\tfrac{1}{m}) \int_0^{z} (1-F^{m*}(t))\, dt$$

$$b(z) = \int_0^{z/m} (1 - F(t))\, dt$$

Areasonable estimate of the expected outdating per period based on an these bounds is

$$\mu(z) = z/m - o\, s(a(z) + b(z))\,.$$

Using this approxnation for the expected outdating per period, one obtain :

$$V(z) = (z + L(z) + \theta u (z) ,$$

$$\hat{V}(x) = -c\ x \qquad \text{and}$$

$$t(z,D) = z-D-\mu(z)$$

It follows then in this case that ignoring constant,

$$W'(z) = c(1- \alpha\ )z + L(z) + (\theta + \alpha c)\ u\ (z)$$

In this case the salvage value assumptioncorresponds to the usual one, namely, that stock remaining at the end of the horizen is solvaged at a return of c per unit and excess demand is made up at a cost of c per unit.

It is also easy to see that $W(z)$ will be convex in z so that it again follows than an (n,s) policy will be optinal for this approximate model. We will denote by (s_1,S_1) the stotionary salution to the system of equations $c_n(x)$;

Computational Results

Having determined two approxmate (s,S) policies, we must compare their performance with that of the optimal policy. Each (s,S) approximation is computed by solving the appropriate functional equations for $C_n(x)$ and interating until the values obtained for (s,S) converged(value interation). policy convergence generally accuned with in less than form periods. For convenience, we assumed that demonds each period were discrete random variables and the reorder and order to points were integers.

The determination of an optimal policy we carried on in the following manner. If the cost of outdating is paid at the time the outdating actually occuns, then the one-period expected cost of ordering, holding, shortage, and outdating , B(x,y) , becomes $B(x,y)= cy+ks(y)+L(x+Y)+\theta\int_0^x(x_1-t)f(t)\,dt$. An optimal policy satisfies the functional equations

$$A_n(x) = \min_{y \geq 0} \left\{ B(x,y) + \alpha\int_0^{\infty} A_{n+1}(t(x,y,u)f(u)\, du \right\} .$$

Charging the outdating cost in this manner is entirely consistent with charging it is we did in section 3. If the term $\theta\int_0^x (x_1-t)f(t)\,dt$ is replaced with $\alpha^{m-1}\theta \int_0^y G_m(u,x)\, du$ above, then the itationary policy obtained by solving for $A_n(x)$ will be indetical. This point is dicussed in detail in a recent note (4) . In order to be consistent θ must be replaced by $\alpha^{m-1}\theta$ in both expressions for W(z) derived in the previous section. We chose this approach for con - verience only. Note that when solving a discrete version of the problem integral rigns must be wins and the minimization is over the set y = 0, I........

Computational experience indicates that the struture of the policy obtained for the one - period model in section 3 is identical to the structure of the optimal that ionary policy . Because the state variable has dimension m^{-1}, the computations quickly become unrealizable as m increases for this reason we consider only the case m = 2.(A test of a few cases with m =3 yielder results that were consistent with those reported here.) For m = 2 the optimal stationary policy is specified by a function $y^*(x)$ that is non increasing in x. As an example, suppose the demand distribution is geometric with mean 10 and C =5,P=10,H=1,θ = 5,K=50, & = 0.25. for this particular case the stationary policy obtained was the following: For $x \leq 0$, $y^*(x) = -x$ and for $x \geq 0$

x =	0	1	2	3	4	5	6	7	8	9	10	11	12	≥ 13
$y^*(x)$ =	24	23	22	21	21	20	19	19	18	18	17	17	17	0

This is certainly not an (s,S) policy. It would be necessary that $x+y^*(x)$ remain fixed in order to be bona. fide (s.S) policy . In this case it was necessary to solve the recurrive equations for $A_n(x)$ interatively for seven periods before policy convergence was ditoined. The approximate policies (s_1,S_1) ans (s_2-S_2) obtained by solving for $C_n(x)$ were, respetively, (3,31) and (12,25) Each calculation required only two periods until policy convergence. Even for m=2 , the difference in computing time is dramatic(about 5 minutes for the exact policy versus 10 reconds for the approximate policy on a POP 10). The computation times required for determining the approximate policies are essentially independent of the rize of m, will that for the optimal policy increases ingnificantly. Computations foa m= 3 require as much as 60 or more minutes of CPU time.

Having determined the optimal policy, $Y^*(x)$, and the approximations(s_1,S_1) and $(s_2;S_2)$ the expected discounted cost associated with each must be computed in order to determine the effectiveness of each opproxition. The average cost per period using the optimal policy and each approximation is ditained from an algorithn developed by Macqueen(5) to an accurcy of 0.01 cost units per period. Rather than report the actual costs obtaived we give the percentage differences between each of the approximations and the optimal policy for the various cases tested . These results are given in Table I .

The particular probability distributions for the one period demond used in the study are (1) uniform on (0, 20), (2) geometric with parameters θ = I/II, and (3) poisson with λ =10 . The distribution parameters are chosen so that mean demand was 10 in each case. These particular distributions were considered in order to compare the effects of skew and unequal variance (the variances are respectively, 33,110 and 10).

An exerimental design was chosen that considered the cost parameters (c,r,θ) at two levels each (c=5 and c=15, r=20 , θ=5 and θ= 20) and the set -up cost k at three levels (k=10,k=50, and k=100) The holding cost was h=1 and the discount factor $\alpha = 0.95$. This particular design was compramise between the desirability of varying as many parameters as possible and the limitations of computing time.

Overall, the approximation(s_2,S_2)based on Chazan an Gal's bounds yielded somewhat lower expected costs, with on overage per - centage differance under 1% for each of the three demand distributions and a maximum evor of 3.7% in all cases tested. The (s_1,S_1) approximation performed slightly better under uniform demand and significantly worse under geometric and poisson demands.

The largest evors invarially occurred for the case K = 100 although it is clear that there is a great deal of interaction between the cost parameters and demand distributions. (For example, for (c,r,θ) beld fixed at (15,20,20) the evors decreased under uniform and geometric demand and increased under poisson demand as K increased .

Percentage Differences Between Optimal Approximate

Costs

Cost case	Uniform		Ceomentric		Poisson		
(c,r,k,e)	$(s_1\ S_1)$	$(s_2\ S_2)$	$(s_1\ S_1)$	$(s_2\ S_2)$	$(s_1\ S_1)$	$(s_2\ S_2)$	)
(5,20,10,5)	0.29	0.12	0.96	0.96	0.03	0.31	Fixed Parameters
(5,20,50,5)	0.11	0.27	0.76	0.05	2.35	0.00	are h= 1, α =0.95
(5,20,100,5)	0.04	0.04	4.03	2.96	1.85	0.93	
(5,40,10,5)	0.52	0.12	2.15	0.75	0.05	0.05	
(5,40,50,5)	0.24	0.24	4.35	0.22	1.15	1.15	
(5,40,100,5)	0.32	1.01	9.01	2.12	4.69	3.70	
(15,20,10,5)	0.36	0.19	0.24	0.24	0.10	0.10	
(15,20,50,5)	0.33	0.31	0.22	0.22	0.28	0.11	
(15,20,100,5)	0.00	0.89	0.28	0.03	2.11	0.22	
(15,40,10,5)	0.28	0.28	0.91	0.91	0.15	0.00	
(15,40,50,5)	0.11	0.57	0.72	0.10	0.30	0.00	
(15,40,100,5)	0.15	0.09	2.20	0.80	2.52	0.00	
(5,20,10,20)	1.13	0.77	0.65	0.80	0.56	0.56	
(5,20,50,20)	0.24	0.24	0.35	1.12	0.37	0.39	
(5,20,100,20)	0.15	0.15	1.18	0.44	2.78	0.87	
(5,40,10,20)	0.16	0.47	1.47	1.15	0.51	0.01	
(5,40,50,20)	0.26	0.26	1.39	0.67	0.61	0.00	
(5,40,100,20)	0.12	0.47	1.51	0.23	2.86	0.14	
(15,20,10,20)	0.48	0.35	0.24	0.41	0.02	0.02	
(15,20,50,20)	0.10	0.26	0.04	0.04	0.25	0.25	
(15,20,100,20)	0.12	0.00	0.51	0.00	0.53	0.53	
(15,40,10,20)	0.56	0.79	0.99	0.99	0.03	1.23	
(15,40,50,20)	0.26	0.19	0.30	0.05	0.10	1.10	
(15,40,100,20)	0.40	0.07	0.47	2.02	0.49	0.49	
Mean	0.31	0.34	1.46	0.72	1.03	0.51	

The performance of each of the approximations depends upon two fac- tors- how closely the form of an (s,S) policy approximates the optimal(s,S) policy. We assumed the optimal value of s were the largest values of x wich that $\overset{*}{y}(x) > o$ which determines suniquely nice m=2. The optimal values of Sare estimated by insepection the function of the $\overset{*}{y}(x)$. For example in the case listed above, $x+\overset{*}{y}(x)$ varied from 24 to 29 as xvaried from 0 to 12. Averaging these values and rounding up we estimate the optimal s value to be 27.

We expect that this method should yield fairly reliable estimates for the optimal (s,S) values when m=2. Estimates obtaived for the optimal (s,S) mal (s,S) values when m = 2.

Estimates obtained for the optimal (s,S) values as well as the values obtained for (s_1,S_1) and (s_2,S_2) are given for each case in Table II. For uniform demand both approximation yielded valies that were consistently with in one unit of each other and were generally within one unit of the optimal values of (s_1,S_1). For geometric demand the value of S_1 overshot the optimal S in most of the cases thus accounting for the evors observed in Table I. the value of S_2 also overshot S in many cases, but the differences were rignificantly smaller. For poissen demand, S_1 was consistently lower than S, while S_2 varied equally above and belows. The degree of error was highest when K was large relative to the other cost parametes. Errors in sappeored to have a more significant effect than errors in s. For example, in cost (15, 40,10,5) under geometric demand, both s_1 and s_2 undershot the optimal s by five units, but resulted in less than 1% error in the expected cost. A similar overshoots occurred in cost case (5,20,100,5) also under geometric demand and resulted in errors of 4.03 % and 2.96% , respectively.

An important point to note is that whenever the approximate and optimal(s,S) values were close, the expected costs were also close. This indicates that the form of an (s,S) policy is an exceellent approximation to the more complex optimal policy ,$Y^*(x)$. In general both approximation yielded expected costs that were within 1% of the optimal with (s_2,S_2) generally giving better results. The largest errors can be anticipated for long - toiled demand distributions and large set- up cost. For which cases, where greater accuracy is needed, testing other values in the neighborhood of the approximate s value can be accomplished by imulating the inventory process. However the degree of precision obtained with the approximate policies derived here should be suitable for most applications.

Table II

Values obtained for (s , S)

Cost case (c,r,k,e)	Uniform			Geometric			Poisson		
	(s_1, S_1)	(s_2, S_2)	Opt $(s_1 S_1)$	$(s_1 S_1)$	$(s_2 S_2)$	Opt $(s S)$	$(s_1 S_1)$	$(s_2 S_2)$	Opt $(s_1 S)$
(5,20,10,5)	(13,17)	(14,17)	(15,17)	(12,19)	(12,19)	(15,17)	(11,14)	(11,15)	(12,14)
(5,20,50,5)	(9,18)	(8,19)	(9,18)	(8,21)	(7,19)	(8,19)	(8,15)	(8,19)	(8,19)*
(5,20,100,5)	(6,19)	(6,19)	(7,19)	(5,26)	(5,25)	(5,20)	(6,18)	(6,22)	(6,20)*
(5,40,10,5)	(16,18)	(16,19)	(17,19)	(19,26)	(17,23)	(22,24)	(13,15)	(13,15)	(13,15)
(5,40,50,5)	(13,19)	(13,19)	(13,20)	(14,31)	(13,25)	(13,27)	(10,15)	(10,15)	(10,18)
(5,40,100,5)	(10,20)	(10,19)	(10,22)	(12,36)	(10,30)	(10,27)	(8,18)	(9,25)	(9,22)*
(15,20,10,5)	(11,15)	(12,15)	(14,16)	(9,13)	(9 ,13)	(12,13)	(11,13)	(11,13)	(11,14)*
(15,20,50,5)	(7,75)	(8,15)	(8,16)	(5,15)	(5,15)	(5,14)*	(8,14)	(7,15)	(8,15)
(15,20,100,5)	(5,17)	(5,19)	(5,17)	(3,16)	(3,15)	(3,14)	(5,15)	(5,19)	(6,18)*
(15,40,10,5)	(15,17)	(15,17)	(17,18)	(14,19)	(14,19)	(19,19)	(12,14)	(13,15)	(13,15)
(15,40,50,5)	(11,18)	(12,19)	(12,18)	(11,21)	(10,19)	(10,20)	(10,14)	(10,15)	(10,15)*
(15,40,100,5)	(9,18)	(9,19)	(9,19)	(8,24)	(8,22)	(8,20)	(8,15)	(8,19)	(8,19)
(5,20,10,20)	(11,14)	(11,15)	(14,15)	(8,12)	(9,13)	(12,12)	(11,13)	(11,13)	(12,14)
(5,20,50,20)	(7,15)	(7,15)	(8,16)	(5,14)	(5,15)	(5,13)	(8,14)	(7,15)	(8,15)
(5,20,100,20)	(5,17)	(5,17)	(5,16)	(2,16)	(3,15)	(2,14)	(6,15)	(5,19)	(6,18)*
(5,40,10,20)	(11,17)	(15,17)	(17,18)	(14,18)	(13,17)	(19,19)	(12,14)	(13,15)	(13,15)
(5,40,50,20)	(11,17)	(11,17)	(12,18)	(10,20)	(9,19)	(10,18)	(10,14)	(10,15)	(10,15)*
(5,40,100,20)	(9,18)	(9,19)	(9,18)	(8,21)	(7,19)	(8,19)	(8,15)	(8,19)	(8,18)
(15,20,10,20)	(10,13)	(11,13)	(13,14)	(7,10)	(7,9)	(10,10)	(11,13)	(11,13)	(11,13)
(15,20,50,20)	(6,14)	(7,15)	(7,14)	(3,11)	(3,11)	(4,11)*	(7,13)	(7,13)	(8,14)
(15,20,100,20)	(4,14)	(4,15)	(4,15)*	(1,13)	(1,11)	(1,11)*	(5,15)	(5,15)	(6,17)*
(15,40,10,20)	(13,16)	(14,17)	(16,16)	(11,15)	(11,15)	(17,17)	((12,14)	(13,13)	(13,14)
(15,40,50,20)	(10,16)	(11,17)	(12,17)	(8,16)	(8,15)	(9,15)	(9,14)	(9,13)	(10,14)
(15,40,100,20)	(8,16)	(9,17)	(9,18)	(6,17)	(6,19)	(6,16)	(8,15)	(8,15)	(8,16)

Note: The optimal values of s are, in general only estimates. The cases where they are exact are marked with an asterisk .

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