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Multi-Objective Linear Programming

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References

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Introduction

This paper attempts to apply methods of mathematical programming to the analiysis of economic problems and the elaboration of algorithm and programms for solving economic models. Such models provide us with analysis of national development which depends on, and interacts with, national economic conditions.

The paper attempts also to present some methods that can be used in the field of macro economic planning, which includes control in addition to planning.

The planning problem, in this paper, is conceived as trying to determins the optimum level of activities, in the light of the given objectives, and within the existing constraints, so formulated the problem of planning is confined to a problem of mathematical programming.

In the following, I will try to show a new approach to the problem of mathematical programming with several objective functions.

Multi-objective formulation is a new way in dealing with many problems, not only in economics, but also in science, industry, etc , Which through their complexity require the simultaneous consideration of several goals. The problem consists in optimizing several objective functions (some of them having to be maximized and others minimized) provided that the variables should verifly a system of linear or non - linear constraints. As arule, it is impossible to find a point in the field of adimisible solutions which should optimize the set of objective functions. Most often, the optimal solution after a functioni is not optimal for the other functions as well, some being even very disadvontageous. For this reason, in the case of several objective functions, the Nation of optimal solution is replaced by the Nations of solution which achieves the best compromise also in the case of single objective function, having as coefficients random variables with a certain distribution function two deterministic objective - functions are naturally appearing which should be taken into account:

The maximizing of the mean value is pursued on the one hand and the mimizing of the objective - function dispersion on the other hand. It is not sufficient to find a solution which maximizes the mean value because if it leads to high dispersion, a parctically in acceptable solution is resulting.

The concsete case we shall refer to is the following.

Let us consider the multi - objective linear progrmming problem as

$$a_{11}x_1 + a_{12}x_2 + \quad x_3 + \dots + a_{1n}x_n \leqslant b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n \leqslant b_2 \qquad (1)$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n \leqslant b_n$$

and

$$X_j \geqslant 0 \quad \text{where} \quad j = 1, \dots, n \qquad (2)$$

under the above constraints find the optimum of

$$F_1 = C_{11}x_1 + C_{12}x_2 + \dots + C_{1r1}x_r$$

$$F_2 = C_{21}x_1 + C_{22}x_2 + \dots + C_{2r1}x_r \qquad (3)$$

$$\vdots$$

$$F_r = C_{r1}x_1 + C_{r2}x_2 + \dots C_{rn}x_n$$

in Matrix Form

$$AX \leqslant b$$

$$X \geqslant 0$$

minum $F = CX$

where

$A = ((a_{ij}))$ $\quad$ 1, ..., m

1, ..., n

is a matrix m x n

$b = (b_1, \ldots, b_m) \in R^m$ is a column vector,

$x = (x_1, \ldots, x_n) \in R^n$ is the unknown Ncolumn vector.

$F = (F_1, \ldots, F_r)$ is a column vector with components representing the objective -functions

$C = ((C_{ij}))$ $\quad i = 1, \ldots, r$ $\quad j = 1, \ldots, n$ is a matrix r X n

We note with

$$D = \{ X = (x_1, \ldots, x_n)' / Ax \leq b, \; x \geq 0 \}$$

set of admissible solutions for the problem.

For fixing the ideas and without the loss of the generality we shall suppose in the following that all fnctions are of maximum.

If this condition is not fulfilled, then we can use the transformation.

$$F(x) = - \max \left\{ - F(x) \right\}$$

Which transform all the functions to be of maximum form.

our problem new is consist in finding that vector

$X^{*} = (x_1^{*}, x_2^{*}, \ldots, X_n^{*}) \in D$ which should be "as good as posible" from the point of View of the set of objective functions F_n $(h = 1, \ldots, r)$.

The definition of X^{*} is under the definition "the best compronuise". In the case of several objective functions. The optinial solution for function is not optimal for the others too-that is why we introduce one of the notions known as a solution "achieving the best compronuise" an "undonunated" solution an efficient solution, on efficacious solution.

In order to define the Vector X^{*} we group the various attempts to find this vector as follow.

1- X^{*} is the vector aptimizing a synthesis-function of the refficiency functions

$$h(F) = h(F_1, F_2, \ldots, F_r)$$

Function b may be defined in various way

a) $h(F_1, \ldots F_r) = \text{optimum} \left\{ F_i(x) \right\}$. $i = 1, \ldots, r$

b) for $r = 2$ it is possible that h

$$h(F_1, F_2) = \frac{F_1(x)}{F_2(x)}$$

Which leads to a usual problem of fractionary programming:

C) $h (F_1,\ldots, F_r) = \sum_{i=1}^{r} \alpha_i (F(x))^{B_i};\ x_i, B_i\ \ 0,$

d) $h (F_1,\ldots, F_r) = -\sum_{i=1}^{r} \alpha_i \exp (-F(x));\ \alpha_i \geqslant 0;$

2- X^* is the vector minimizing on optmality criterion of the form

$$\Phi (X^*) = \min h (\Upsilon_1(x - X), \ldots, \ldots \Upsilon_r (x - x) \mid x \in D$$

Where: $X_j = (x_{1j},\ldots, x_{nj})$ is the optimal solution of the objective function F_j and Υ_r is a function of distance between Vector $x \in D$ and the optimal solution X_k of function F_k.

The concrete selection of functions h and Υ_k allows to obtain porticular cases of expression (4) for example.

a) $h (x) = \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{n} (x_j - x_{jk})\ ;\quad \alpha_k \geqslant 0$

b) $h (x) = \sum_{k=1}^{r} \alpha_k \sum_{j=1}^{n} \mid x_j - x_{jk} \mid ;$

c) $h(x) = \max \Psi_k (x - X_k)$; $k = 1, \ldots, r$

d) $h(x) = \sum_{k=1}^{r} \Psi_k (x - X_k)$

e) $h(x) = \prod_{k=1} \Psi_k (x - X_k)$

3- X^* is the Vector belonging to a set of efficient points which is definded as follow. A point $x^0 \in D$ is said to be efficient if and only if dose not exist onther point $x \in D$ so that $F_h(x) \geq F_h(x^0)$, $h = 1 = 1, \ldots, r$ and for at least one h_0 we should have $F_{h0}(x) > F_{h0}(x^0)$, supposing that all functions are of maximum.

In other words. x^0 is efficient if there is no point x which should improve at least one function, while the others remain unchanged.

The notion of efficiently optimal solution is playing an important part in economy in the game theory, in the statistical decisions theory and generally, in any problems of decisions with several in compareable criteria.

As we shall see later the determining of the efficient points includes most of the optimizing methods in mathematical progrmming.

4- X^* is an optimal solution obtaimed by ordering criteria and which is attained as follows:

we solve r problems of mathematical programming, eahh time restricting the field D by turning into constraints the optimal solutions obtained by solving a certain problem with a single functions, More exactly, we deremine successively we sets:

$$D_0^* = D$$

$$D_1^* = \left\{ x \mid F_1(x) = \underset{y \in D_0^*}{\text{optimum}} F_1(y) \; ; x \in D_0^* \right)$$

$$D_2^* = \left\{ x \mid F_2(x) = \underset{y \in D_1^*}{\text{optimum}} F_2(y) \; ; x \in D_1^* \right\}$$

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$$D_k^* = \left\{ x \mid F_k(x) = \underset{y \in D_{k-1}^*}{\text{optimum}} F_k(y) \; ; x \in D_{k-1}^* \right\}$$

$$D_r^* = \left\{ x \mid F_k(x) = \underset{y \in D_{r-1}^*}{\text{optimum}} F_r(y) \; ; x \in D_{r-1}^* \right\}$$

The solving of the problem with several objective functions represents the derermining of one of several points of sets D_r^*. Obviously this set D_r^* is closely linked with the order of functions. Generally, different sets are corresponding to two different orders.

5- X^* is a point of the field of admissible solution which is obtained by a seeking method according to certain criteria

6- X^* belongs to a set of properly efficient point which is defined as follows

A point x^* is a properly efficient solution if it is efficient, and if there is a scalar $M > 0$ such that $F_i(x) > F_i(x^*)$ implies

$$\frac{F_i(x) - F_i(x^*)}{F_j(x^*) - F_j(x)} \leq M$$

for some j weith $F_j(x) \leq F_j(x^*)$ In the linear case the two sets coincide.

2- Relationschips Between various methods of soluing the problems of Multi-objective Functions:

We note with $D^{**} \subset D$ the set of efficient solutions and with D_r^* the set of optimal solutions obtained by ordering the criteria.

The connection between the two sets can be seen from the following

1- Any optimal solution obtained by ordering the criteria is an efficient point of these criteria, namely: $D_r^* \subseteq D^{**}$ to prove this let us suppose, against all reason, that $D_r^* \not\subseteq D^{**}$ that is there is a point $x' \in D_r^*$, so that $x' \notin D^{**}$. point x' not being efficient it results that there exists anther point $x'' \subseteq D$ with the characteristic that $F_k(x'') \geq F_k(x')$, $k = 1,\ldots,r$ and for at least on index 1 (one of indicies $1,\ldots,r$) the inequality is strict.

$$F_1(x'') > F_1(x') \qquad (5)$$

Consider $k = 1$. As $x' \in D$, and $D_r^* \subset D_1^*$, it results that $x' \in D_1^*$ thus x' is a maximum point of function F_1.

As we have $F_1(x'') \geq F_1(x')$ it results that this ratio can be verified only with the equal sign, that is

$$F_1(x'') = F_1(x')$$

For $k = 2$. resulting again will be $F_2(x') = F_2(x'')$ thus $x'' \in D_2^*$. in a similar way we obtain:

$$F_k(x'') = F_k(x') \quad \text{for any } k = 1,\ldots,r$$

which condradicts (5).

J. saska suggests for solving the problem of linear programming with several objective functions a method consisting in finding vector $X^* \in D$ which minimizes the function

$$h(x) = \max_{1 \leq h \leq r} \frac{\left| X_h - F_h(x) \right|}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} \qquad (6)$$

Where

$$X_h = (\text{optimum})\, F_h(x). \quad x \in D$$

In this way, Vector $X^* = (x^*_1, \ldots, x^*_n) \in D$ has the characteristic that it is the least distant modulus from the byerplanes determined by the r objective. Functions that is

$$\max_{1 \leq h \leq r} \frac{\left| X_h - F_h(X^*) \right|}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} = \min_{x \in D} \left[\max_{h=1\ldots,r} \frac{\left| X_h - F_h(x) \right|}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} \right. = \delta \qquad (7)$$

Function h (x) is convex (thus the existence of the minimum Value is ensured) but non. linear, a fact which makes difficult the effective mumberical determination of optimal solution. This non linear restriction can be linearized as

$$\begin{aligned} Ax &\leq b \\ x &\geq 0 \\ V^{(k)}x + x_{n+1} &\geq \tilde{X}_k \\ V^{(k)}x - x_{n+1} &\leq \tilde{X}_k \\ \min \Upsilon(x) &= x_{n+1} \end{aligned} \qquad (8)$$

Where

$$V^{(k)} = (V_{ki}) = \left(\frac{c_{ki}}{\sqrt{\sum_{j=1}^{n} c_{hj}^2}} \right.$$

$$, k = 1,\ldots, r$$
$$i = 2,\ldots, n$$

$$\hat{X}_k = \frac{X_k}{\sqrt{\sum_{j=1}^{n} C_{kj}^2}} \qquad k = 1,\ldots, r$$

and $X_n + 1$ is a complementary variable so that

$$\frac{\left| X_k - F_k(x) \right|}{\sqrt{\sum_{j=1}^{n} C_{kj}^2}} \leqslant X_{n+1}; \quad k = 1,\ldots,r \qquad (9)$$

2- The optimal solution ($\hat{x}$, $\hat{x}_{n+1}$) of the programming problem (8) is equal to the optimal solution (x^*, δ) of the problem of non linear programming having the objective function (6). to prove this let us suppase that we have: $\min \Upsilon(x) = \hat{x}_{n+1} =$

$$= \max_{1 \leqslant k \leqslant r} \frac{\left| X_k - F_k(\hat{x}) \right|}{\sqrt{\sum_{j=k}^{n} C_{kj}^2}}$$

$$= \max_{1 \leqslant k \leqslant r} \left| \hat{X}_k - V^{k1} \hat{x} \right| \geqslant \min_{x} \left\{ \max \left| \hat{X}_k - V^k \right| \right\}$$

$$= \min_{x} \left\{ \max_{1 \leqslant k \leqslant r} \frac{\left| X_k - F_k(x) \right|}{\sqrt{\sum_{j=1}^{n} C_{kj}^2}} \right.$$

$$= \max_{1 \leq k \leq r} \frac{\left| X_k - F_k (x^*) \right|}{\sqrt{\sum_{j=1}^{n} C^2_{kj}}} = \delta \quad (10)$$

Ratio (10) shows that

$$\hat{x}_{n+1} \geq \delta \quad (11)$$

As we can write that min $x_{n+} = \hat{X}_{n+1}$ also resulting is the inequality:

$$\hat{x}_{n+1} \leq \delta \, .$$

Resulting from (10) and (12) is

$$x_{n+1} = \max_{1 \leq k \leq r} \left| \tilde{X} - V^{(k)} \hat{x} \right|$$

$$= \max_{1 \leq k \leq r} \left| \tilde{X}_k - V^{(k)} x^* \right| = \delta \quad (12)$$

J. Saska gvies anther method for solving the multiple criteria problem considering instead of function 6, Function

$$h_1 (x) = \max_{1 \leq k \leq r} \frac{\left| X_k - F_k (x) \right|}{X_k}$$

Which must be minimized. Following the some idea of papers of t mm, M, I we onsider that

$$\psi_i(x - X_i) = F_i(x_i) - F_i(x)$$

and that
$$h\left[\psi_1(x - X_1), \ldots, \psi_r(x - X_r)\right] = \sum_{j=1}^{r}\left[F_i(x_i) - F_i(x)\right]^2 = \phi(x)$$

S. . Haung considers the problem of determining the vector x^* which minimizes function $\phi(x)$ and defines it as a vector solving the problem with several objective functions. Thus x^* minmizes the sum of quadrates of deviations from absolute optimal values, it is proved that x^* is on efficient point.

Theorem: Any vector x^* minimizing on the field D function

$$\phi(x) = \sum_{i=1}^{r}\left[F_i(x_i) - F_i(x)\right]^2 \qquad (13)$$

. an efficient point.

proof : let us admit that x^* minimizes function $\phi(x)$ that is, for any ather point x we have

$$\sum_{i=1}^{r}\left[F_i(x_i) - F_i(x_i^*)\right]^2 \leq \sum_{i=1}^{r}\left[F_i(x_i) - F_i(x)\right]^2 \qquad (14)$$

but x^* is not efficient, thus there is a point X with the characteristic:

$$F_i(x^*) \leqslant F_i(\bar{x}) \qquad i = 1,\ldots,k$$

(14)

$$F_i(x^*) < F_i(\bar{x}) \qquad i = k+1,\ldots,r$$

on the other hand

(15)

$$F_i(X_i) - F_i(x^*) \geqslant 0 \quad i = 1,\ldots, r$$

$$F_i(X_i) - F_i(\bar{x}) \geqslant 0 \quad i = 1,\ldots, r \qquad (16)$$

From $F_i(X_i)$ we subtract ratios (14)

and taking into account (15) and (16) we have

$$F_i(x_i) - F_i(x_i^*) \geqslant F_i(X_i) - F_i(\bar{X}) \geqslant 0$$

(17)

$$F_i(X_i) - F_i(x^*) \geqslant F_i(x_i) - F_i(\bar{x}) \geqslant 0$$

for $i = 1,\ldots, k$

from 17 by squaring and totaling according to $i = 1,\ldots, r$ on both sides we obtain

$$\sum_{i=1}^{r}\left[F_i(x_i) - F_i(x^*)\right]^2 > \sum_{i=1}^{r}\left[F_i(x_i) - F_i(\bar{x})\right]^2$$

which contradicts (14) . Thus x^* is efficient consider now

$$\Upsilon_i(x - X_i) = F_i(X_i) - F_i(x)$$

and

$$h\,(\Upsilon_1(x - X_1) \ldots \Upsilon_r(x - X_1) = \max_i \left[F_i\,(X_i) - F_i\,(x) \right] = \Upsilon(x)$$

it can be agreed that vector x^* which minimizes function $\Upsilon(x)$ should be the solution of problem with several objective functions.

Point x^* minimizes the maxinum deviation from the optimal value of the objective functions.

Theorem : Any solution x^* which minimizes function

$$\Upsilon\,(x) = \max_i \left[F_i\,(X_i) - F_i\,(x) \right]$$

on the field D is an efficient solution

Proof : Indeed, if x^* is not efficient there will exist - x' so that $F_i\,(x') \geqslant F_i\,(x^*)$ and at least one of the inequalities is strict.

Hence

$$F_i\,(X_i) - F_i\,(x') \leqslant F_i\,(X_i) - F_i\,(x^*)$$

that is

$$\max_i \left[F_i\,(X_i) - F_i\,(x') \right] < \max_i \left[F_i(X_i) - F_i(x^*) \right]$$

Which contradicts the hypothesis that x^* minmizes function $\Upsilon\,(x)$.

3- Gool Programming

In tackling so for the problem with several objective functions we have admithed that r problems of mathematical programming are solved frist and then, in keeping with the optimal values found, a synthesis function is built.

We shall present anther group of methods known in literature under the name of goal - programming we consider a vector $\overline{F} = (\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_r)$ whose components, represent the levels to be attained by the objective functions. For a $x \in D$ there will be more or less deviations from these values and the problem is to minimize a function measuring the distance between vector $\overline{F}$ and the vector whose components represent, the possible values of the objective functions.

We shall consider the vectorial space n - dimensional R^n endowed with the norm $\| \cdot \|$ classically defined. consider two points $q = (q_1, \ldots, q_n)$ and $r = (r_1, \ldots, r_n)$ and $d(q, r) = \| q - r \|$ we shall further use instead of the distance between q and r, as a measure of opproach between $\overline{F}$ and F.

The best known norm is norm L_p or the Hölder norm.

$$\| x \| = \left(\sum_{i=1}^{n} | x_i | \right)^{\frac{1}{p}}, \quad p \geq 1$$

In particular cases the following norms are obtained

$$P = 1$$

$$\|x\|_1 = \sum_{i=1}^{n} |x_i| ;$$

$$p = 2$$

$$\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{\frac{1}{2}}$$

$$p = \infty$$

$$\|x\|_\infty = \max_i \left\{ |x_i| \right\}$$

Norm $\| \cdot \|_2$ represents the Eucliden distance If we take the norm $F - \bar{F}$ the following cases are obtained which solve the goal programming problem :

Case 1

$$\min_x \left\{ \|F - \bar{F}\|_p = \left[\sum_{i=1}^{n} |F_i - \bar{F}_i|^p \right]^{\frac{1}{p}} \;\middle|\; Ax = b ,\ x \geqslant 0 \right\} .$$

Case 2

$$\min_x \left\{ \|F - \bar{F}\|_1 = \sum_{i=1}^{r} |F_i - \bar{F}_i| ,\ \middle|\ Ax = b ,\ x \geqslant 0 \right\}$$

Case 3

$$\min_x \left\{ \| F - \bar{F} \|_2 = \left[\sum_{i=1}^{n} \left| F_i - \bar{F}_i \right|^2 \right]^{\frac{1}{2}} \;\middle|\; Ax = b \,,\; x \geqslant 0 \right\}$$

Case 4

$$\min \left\{ \| F - \bar{F} \|_\infty = \max \left| F_i - \bar{F}_i \right| \;\middle|\; Ax = b \,,\; x \geqslant 0 \right\}$$

taking into account Hölder's inequalities norm $\| \cdot \|_p$ is a convex function.

We also note that other forms of case 3 as

$$\min_x \left\{ \| F - \bar{F} \|_2 = \left[\sum_{i=1}^{r} (F_i - \bar{F}_i)^{2} \right] \middle| Ax = b \,;\; x \geqslant 0 \right\}$$

or

$$\min_x \left\{ \| F - \bar{F} \|_2 = \left[\sum_{i=1}^{r} \alpha_i (F_i - \bar{F}_i)^2 \right] \middle| Ax = b \quad x \geqslant 0 \right\}$$

Wher α_i are the importance coefficients of functions F_i.

The minimizing of norm $\| F - \bar{F} \|_p$, for $p > 2$ leads to nonlinear programming problems where as the minimizing of norms $\| F - \bar{F} \|_1$ and $\| F - \bar{F} \|_\infty$ can be done by simplex method of the linear programming.

Indeed, let us consider one of the cases, say case 2 this case was studied by A.Hamza, A.charnes and W.W coopers. We note that with $d_k^+(x)$ and $d_k^-(x)$ for each function, the deviations with plus or minus, from values $\overline{F}_k$. The problem is to minimize the total of these diviations

Case 2 becomes

$$\min \sum_{k=1}^{r} d_k^+ (x) + d_k^- (x)$$

$$F_k (x) + d_k^- - d_k^+ = \overline{F}_k \quad , k = 1,\dots,r \qquad (18)$$

$$d_k^+ (x) \ , \ d_k^- (x) \geq 0$$

$$x \in D$$

As for a function F_k the deniation from $\overline{F}_k$ in a point takes place only in one divection, it will result that if $d_k^+ > 0$ in this case $d_k^- = 0$ and reciprocally if $d_k^- > 0$ in this case $d_k^+ = 0$

Note with e a column vector having relement, all equal to 1, to I the unit matrix of the order r and to d^+ and d^- the component vectors d_i^+ and d_i^- 18 can be written in the form

$$\min \left[ed^+ + ed^- \right]$$

$$F(x) - Id^+ + Id^- = \overline{F}$$

$$Ax = b$$

$$x, d^+, d^- \geq 0$$

it can be proved that any optimal solution of problem (18) is an efficient solution.

(see A. Hamza)

Case 4 can by put in the following linear form (see Zubouitk S.I.V.A. and L.I. vdeena)

$$\min \lambda$$

$$-\lambda \leq \bar{F}_i - F_i \leq \lambda$$

$$Ax = b$$

$$x, \lambda \geq 0$$

As regards to case 3 it can be solved by the method of generalized inverses . The solution is

$$x = R^{+}F + R^{0}B$$

Where R^{+} having the order n x r is the generalized inverse of R of the degree q, R^{0} is a matrix nx (n - q) with the characteristics that $RR^{0} = 0$ and B is an arbitrary vector with n - r components. R^{0} and B are deremind from the other conditions an x, for example from $x \geq 0$ that is from $R^{+}F + R^{0}B \geq 0$ we deremine R^{0}, and from $Ax = b$ that is $AR^{+}\bar{F} + AR^{0}B = b$ we determine B.