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COMPROMISE SOLUTIONS IN MULTIOBJECTIVE LINEAR PROGRAMMING PROBLEMS

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ABSTRACT

This paper discusses some problems of multiple objective linear programming /MOLP/ solving and MOLP optimal solutions finding. MOLP model is formulated as in /8/.

After concise MOLP historical review the problem definition and optimal solutions finding, in relation to information on objective functions, is presented in section 2. The division of information is performed as follows:

I – Minimal levels for goals are assumed,

II– Goals hierarchy is present,

III–Utility function is given and calculated.

In section 3 MOLP problem without goals hierarchy is dealt with. As its optimal solutions one takes these, satisfying so called "Pareto Optimum", i.e. efficient solutions. Some basic properties of efficient solutions ~~set~~ $\mathcal{E}$ are presented.

Section 4 is devoted to the problem of compromise solutions, which are generated, when set $\mathcal{E}$ is too big. To do so, optimal solutions of problems a/ /16/, b/ /21/ – /22/, c/ /23/ – /24/, d/ /25/ – /26/ can be used. Some of these problems are developed by the author, the others are constructed on the basis of recently published research.

Solution procedure of problem /27/, proposed by M.Zeleny, which generates compromise set is discussed in the last section.

1. INTRODUCTION

It is well known that quite frequently the decision taken is not evaluated on the basis of single value, but on the basis of several values, belonging to the set of indices. Then, we have multiple indices or multicriteria evaluation of a decision. Single valued evaluation is usually narrow quality description. Multiple indices evaluation enables to perform more complex characterisation and also decision making in modern economy is carried out by several goals achievement.

It is interesting that multiple criteria decision making was discussed simultaneously with quantity formulation of economy theory. V. Pareto introduced in 1896 [12] the notion of optimal decision with respect to several criteria, widely known as "Pareto optimum". During the early years of linear programming theory T.C.Koopmans [9] and H.W. Kuhn, A.W.Tucker [10] referred to the problem of optimal solutions finding in multicriteria mathematical programming models.

One can say that a significant research on optimal solutions and its definition in linear programming models with several objective functions was started in 1965. This time, a problem was called multiple objective linear programming /MOLP/ or multicriteria linear programming problem.

It should be stressed that first multiple criteria LP formulation can be encountered in [1], where A. Charnes and W. Cooper formulated a goal programming problem.

Feasible decisions set $\mathcal{X}$ is given by the following constraints:

$$\begin{cases} \mathbb{A}x = \mathbb{b} & (\mathbb{b} \geqslant \mathbb{0}) \\ x \geqslant \mathbb{0}, & x \in R^n \end{cases} \qquad /1/$$

where matrix $\mathbb{A}$ has dimensions m x n.
Each decision $x \in \mathcal{X}$ has to satisfy t different goals, where goals satisfaction by x are measured by linear functions

$$f_i(x) = \mathbb{c}_i x \qquad /i = 1, \ldots, t/. \quad /2/$$

Value d_i / i = 1, ..., t/ of a required accomplishment is assigned to every goal.

In goal programming we look for the optimal solution of the problem

$$\min \left\{ \sum_{i=1}^{t} p_i \left| \mathbb{c}_i x - d_i \right| \,\middle|\, x \in \mathcal{X} \right\}, \qquad /3/$$

where p_i are deviations comparability coefficients.
It is known that for $p_i > 0$ problem /3/ can be formulated as a corresponding LP problem:

$$\sum_{i=1}^{t} p_i (y_i^+ + y_i^-) \longrightarrow \min \qquad /4/$$

s.t.

$$\begin{cases} \mathbb{A}x & = \mathbb{b} \\ \mathbb{C}x - \mathbb{y}^+ + \mathbb{y}^- & = \mathbb{d} \\ x \geqslant \mathbb{0}, \quad \mathbb{y}^+ \geqslant \mathbb{0}, \quad \mathbb{y}^- \geqslant \mathbb{0}, \end{cases} \qquad /5/$$

where

$$\mathbb{C} = \begin{bmatrix} \mathbb{c}_1 \\ \vdots \\ \mathbb{c}_t \end{bmatrix}, \quad \mathbb{d} = \begin{bmatrix} d_1 \\ \vdots \\ d_t \end{bmatrix}; \quad \mathbb{y}^+ = \begin{bmatrix} y_1^+ \\ \vdots \\ y_t^+ \end{bmatrix}, \quad \mathbb{y}^- = \begin{bmatrix} y_1^- \\ \vdots \\ y_t^- \end{bmatrix}.$$

Problem /3/ solving gets complicated when different $/p_i^+/$ values are assumed for "above" d_i deviations. and different $/p_i^-/$ for "below" d_i deviations. In such a case substitution of /3/ by problem /4/ - /5/ is possible [15] if $p_i^- \geqslant -p_i^+$ /i = 1, ..., t/.

Assumption of a specific goals achievement levels is one among different approaches to multiple criteria decision making. It is usually performed by the inclusion of some inequalities to definition of ~~the feasible~~ decisions set, and not as in /3/ by the inclusion of additional information to the objective functions. We shall discuss this problem further on.

2. Formulation of MOLP problem and its properties.

2.1. Formulating MOLP problem we assume:

1° - The set $\mathcal{X}$ of feasible decisions defined by constraints / 1 / is bounded;

2° - Every decision $x \in \mathcal{X}$ is evaluated on the grounds of t different goals by the vector - function /2/ ;

$$F(x) = C x . \qquad /6/$$

3° - In a decision making process we want to maximize every $f_i(x)$ separately on the feasible set $\mathcal{X}$, so

$$\forall_{1 \leq i \leq t} \ \exists_{x_i^0 \in \mathcal{X}} \quad c_i x_i^0 = \max \{ c_i x \mid x \in \mathcal{X} \} = M_i \qquad /7/$$

and x_i^0 is an absolute optimal solution for goal "i".

Taking into account the assumptions 1° - 3° we shall define MOLP problem as:

$$\max \{ F(x) \mid x \in \mathcal{X} \} . \qquad /8/$$

Function F /x/ is a linear mapping of R^n space into R^t, and because $\mathcal{X} \subset R^n$ and $\mathcal{F}(\mathcal{X}) = \{ F(x) \mid x \in \mathcal{X} \} \subset R^t$, so R^n - decision space,

R^t - criteria space.

Linear mapping guarantees that

a/ if $\mathcal{X}$ is a convex polyhedron, than F /$\mathcal{X}$/ is also a convex polyhedron;

b/ the ~~inverse~~-image of F /$\mathcal{X}$/ vertex is at least one vertex of the set $\mathcal{X}$.

In single - objective LP, which is a special case of MOLP, for every two decisions $x_1 \neq x_2$, belonging to $\mathcal{X}$, the following inequalities, ordering their performance, hold:

$$F/x_1/ \geqslant F/x_2/ \quad \text{or} \quad F/x_1/ < F/x_2/.$$

In MOLP problem it has not to occur /see fig 1a, 1b/ what implies serious difficulties ~~with~~ the definition of optimal solution in problem /8/. As we shall see, all the approaches will be to a certain extend based on a single criteria optimization.

Optimal solution definition largely depends on information concerning problem /8/ objective functions, which we have on hand. We distinguish four kinds of such information:

I/ minimal levels of goals satisfaction are assumed,

II/ a strict goals hierarchy is present,

III/ an utility function U/F/, representing all the goals is defined on set F/$\mathcal{X}$/,

IV/ all problem /8/ goals are equally important.

2.2. Usually information on minimal goal performance are linked with one of the remaining groups of information and are included in set $\mathcal{X}$ definition /as inequalities.

$$\mathbb{C}x \geqslant \mathbb{d}.$$

Let us denote

$$\mathcal{J} = \{ x \mid \mathbb{C}x \geqslant \mathbb{d},\ x \geqslant 0 \}.$$

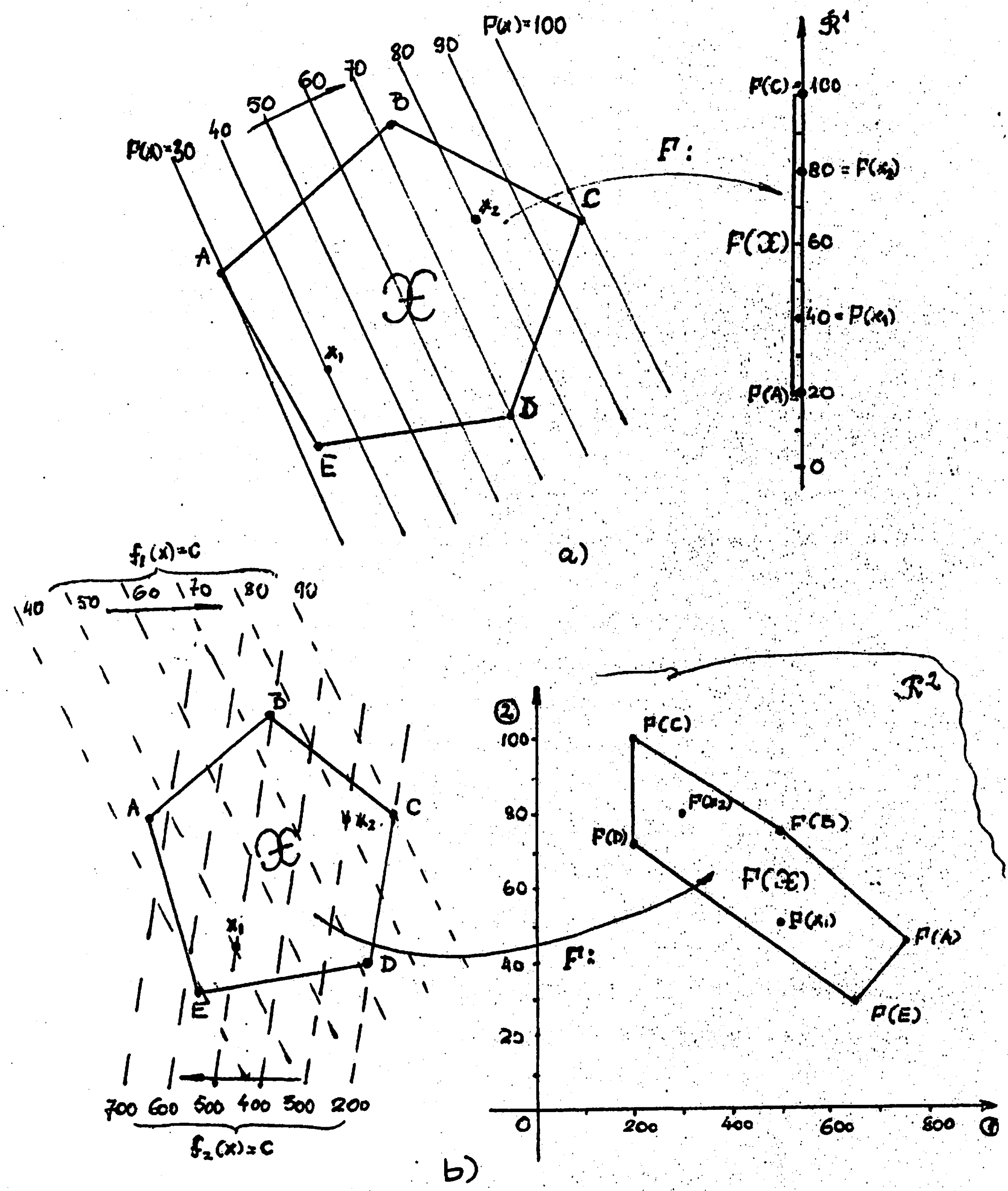
F(x)=100
90
80
70
60
50
40
F(x)=30
A
B
C
D
E
x₂
x₁
F:
R¹
F(C)=100
80 = F(x₂)
F(X)
60
40 = F(x₁)
F(A)=20
0
a)
f₁(x)=c
40
50
60
70
80
90
A
B
C
D
E
x₁
x₂
700 600 500 400 300 200
f₂(x)=c
R²
②
100
80
60
40
20
0
200
400
600
800
①
F(C)
F(x₂)
F(B)
F(D)
F(X)
F(x₁)
F(A)
F(E)
F:
b)

Figure 1

Now we consider when $\mathbb{J} \cap \mathfrak{X} = \emptyset$. In this case problem /8/ should be replaced by searching for such $x \in \mathfrak{X}$, which has possibly "the best" location with respect to $\mathbb{J}$ /see fig.2/. When $\mathbb{J} \cap \mathfrak{X} \neq \emptyset$ problem /8/ is consistent, and its optimal solution is defined in relation to other information on objective functions $f_i/x/$.

We check for sets $\mathbb{J}$ and $\mathfrak{X}$ disjunction by solving problem

$$[1]^T y \longrightarrow \min \qquad /9/$$

s.t.

$$\begin{cases} A x = b \\ C x - y = d \\ x \geqslant 0 \ , \ y \geqslant 0 . \end{cases} \qquad /10/$$

Let (x^0, y^0) denotes optimal solution of /9/ - /10/.
Sets $\mathbb{J}$ and $\mathfrak{X}$ are disjoint iff $[1]^T y^0 > 0$.
If sets $\mathbb{J}$ and $\mathfrak{X}$ are disjoint, two approaches are possible[1].

A/ Unit loss weights p_i^- resulting from not attainement of required levels d_i are given. Then we are looking for the optimal solution of the problem

$$\sum_{i=1}^{k} p_i^- \ y_i^- \longrightarrow \min$$

s.t.

conditions /5/.

B/ It is possible to overtake sets $\mathbb{J}$ and $\mathfrak{X}$ disjunction by changing b_1 - part of b vector, into $b_1 + w$. This change implies costs $k w$. The amount of change can be limited: $w \leq \beta$, the cost of change can be limited as well: $k w \leq \alpha$. As a result we solve:

$$k w \longrightarrow \min \qquad /11/$$

s.t.

1/Here I use some proposals presented by dr E.Konarzewska at seminar held in Research Institute for the Management Organization and Development, Warsaw, in 1976.

$$\begin{cases} \mathbb{A}_1 x - w \leq \mathbb{b}_1 \\ \mathbb{A}_2 x = \mathbb{b}_2 \\ \mathbb{C} x \geqslant \mathbb{d} \\ \mathbb{k} w \leq \alpha \\ x \geqslant \mathbb{0}, \quad \mathbb{0} \leqslant w \leq \beta \end{cases} \tag{12}$$

If constraints /12/ are consistent and $\min \mathbb{k} w < \alpha$, then it is possible to use problem /8/ with feasible set defined by constraints /12/ while computing final optimal solution.

2.3. Let the objective functions ordering represent existing goals hierarchy. Then as an optimal solution of problem /8/ we take this one of the last problem, from consecutively solved "t" LP problems.

$$\max\left\{\mathbb{c}_i x \mid x \in \mathcal{X}_i\right\} \qquad (i=1,2,\ldots,t) \tag{13}$$

where

$$\mathcal{X}_1 = \mathcal{X}, \quad \mathcal{X}_i = \left\{x \mid x \in \mathcal{X}_{i-1}, \ \mathbb{c}_{i-1} x \geqslant (1-q_{i-1})\bar{M}_{i-1}\right\} \qquad (i=2,\ldots,t)$$

$$\bar{M}_i = \max\left\{\mathbb{c}_i x \mid x \in \mathcal{X}_i\right\} \qquad (i=1,\ldots,t-1)$$

$0<q_i<1$ - feasible percentage deviation of f_i function from $\bar{M}_i$ value; values q_i can be different. The main disadvantage of this approach lies in the fact that apart from goals hierarchy, final optimal solution may be closer to absolute optimum of last goals, than the first ones/see fig.3/

To avoid such situation it is possible to start the procedure by solving auxiliary problem for first "s" objective functions /s < t/

$$v \longrightarrow \min$$

s.t.

$$\mathcal{X}(v) \sim \begin{cases} \mathbb{A}x = \mathbb{b} \\ \mathbb{C}_i x \geqslant (1-v) M_i & (i=1,\dots,s) \\ x \geqslant \mathbb{0}, \quad 0 \leq v \leq 1 \end{cases}$$

With optimal solution $/x_0, v_0/$ the set $\mathcal{X}(v^o) \subset \mathcal{X}$ defines an area, where every of s distinguished objective functions varies from its absolute maximum less than v_0 %. For some $\Delta v > 0$ we compute $\bar{M}_1$ in set $\mathcal{X}_1 = \mathcal{X}(v_0 + \Delta v)$ and for i = s+1,...,t we use problem /13/, taking $\mathcal{X}_s = \mathcal{X}_1$ and $\bar{M}_s = \bar{M}_1$.

2.4. Utility function construction is widely discussed in the literature, with an extensive survey given by Fishburn in [7], so we shall not describe it here. Existence of a utility function $U[F/x/]$ enables to replace multiple criteria problem /8/ by single criteria mathematical programming problem:

$$\max \left\{ \mathcal{U}[F(x)] \,\middle|\, x \in \mathcal{X} \right\}. \qquad /14/$$

When utility function $U[F/x/]$ is a linear function $f_i /x/$, i.e.

$$U/x/ = \sum_{i=1}^{t} p_i \cdot \mathbb{C}_i x$$

than problem /14/ can be contracted to LP problem: $\max\left\{ \left(\sum_{i=1}^{t} p_i \mathbb{C}_i\right) \cdot x \,\middle|\, x \in \mathcal{X} \right\}$

3. MOLP problem with equally important objective functions.

Every vector $\bar{x} \in \mathcal{X}$ satisfying Pareto optimum in reference to objective function /2/ is taken as an optimal solution of problem /8/. Because of assumptions 1°, 2° stated with problem /8/ definition, vector $\bar{x}$ satisfies Pareto optimum condition - and is its efficient solution, iff

$$\nexists_{x \in \mathcal{X}} \quad F(x) \gneqq F(\bar{x}). \qquad /15/$$

Let $\mathcal{E}$ denotes efficient solutions set of problem /8/. Obviously $\mathcal{E} \subset \mathcal{X}$.

The following theorems dealing with efficient solutions are true /for proofs see for example [4], [16], [17]./

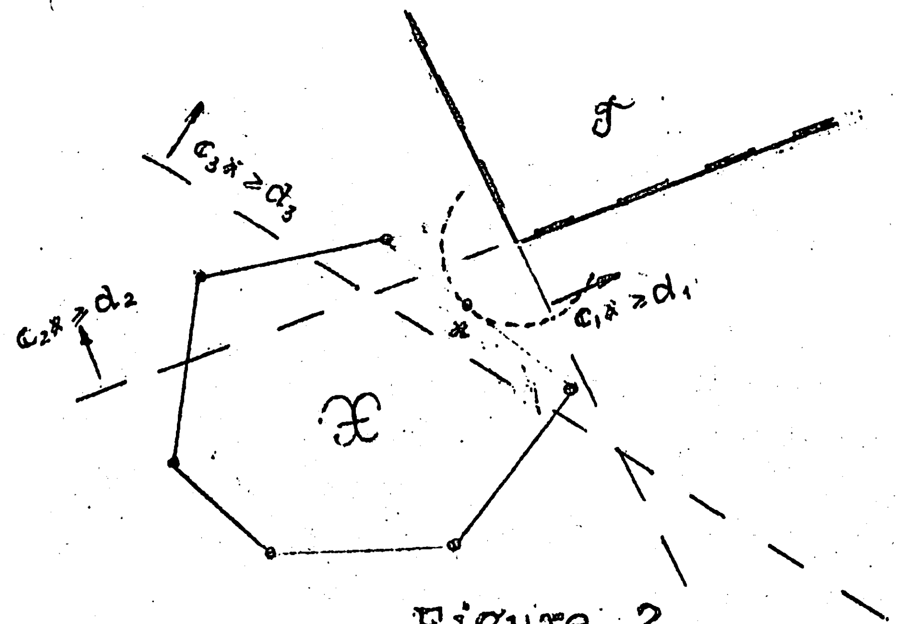

Figure 2

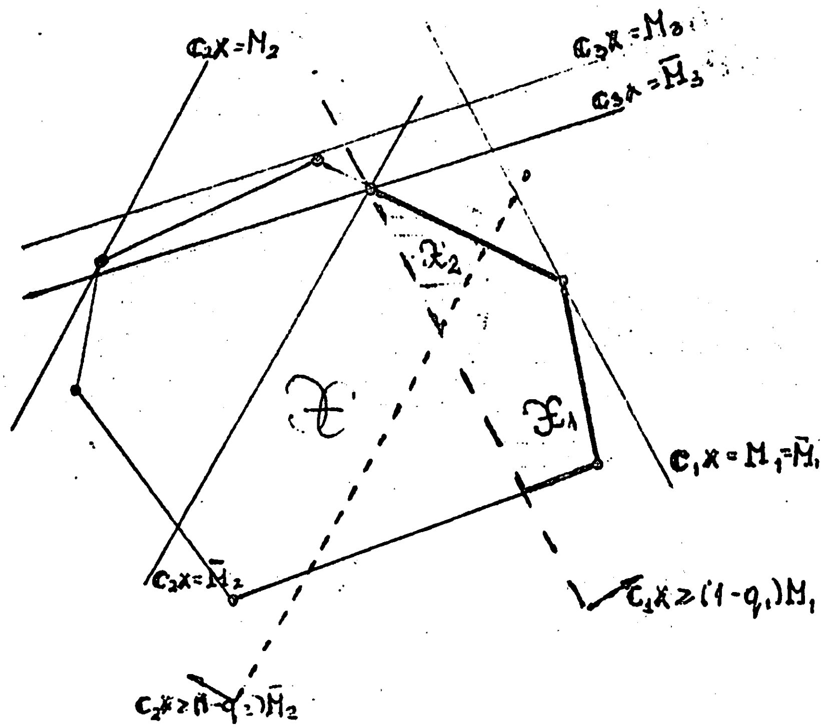

Figure 3

Theorem 1

If x_i^o is the only optimal solution of problem /7/, then $x_i^o \in \mathcal{E}$
If problem /7/ has an infinite number of optimal solutions, then at least one is efficient.

Let $\mathcal{E}_b$ denotes basic efficient solutions set for problem /8/.

Theorem 2

If $\mathcal{E} \neq \emptyset$, then $\mathcal{E}_b$ has a finite number of elements.

Theorem 3

When set $\mathcal{E}$ is bounded, every $\bar{x} \in \mathcal{E}$ can be presented as convex combination of some elements from $\mathcal{E}_b$.

Theorem 4

$\mathcal{X} - \mathcal{E}$ is a convex set.

Theorem 5

The feasible vector $\bar{x} \in \mathcal{X}$ is also efficient one $(\bar{x} \in \mathcal{E})$, iff there exists such vector $p \in R^t$, that $p > 0$, $[1]p = 1$ and

$$p^T C \bar{x} = \max \{ p^T C x \mid x \in \mathcal{X} \}. \qquad /16/$$

On the basis of theorem 3, M. Zeleny [17] developed an efficient procedure for set $\mathcal{E}$ determination. It is two-stage algorithm. In the first stage set $\mathcal{E}_b$ is generated by adequately modified simplex method. In the second stage, sets of convex combinations of $\mathcal{E}_b$ elements are created. The sum of these sets defines $\mathcal{E}$. The other modification of simplex method /as a matter of fact two modifications/ is presented by Evans and Steuer [5], [6]. It was programmed and is successfully used in practice. I believe it completes the list of actually the most efficient procedures of MOLP problems solving.

Apart from the above, theorem 5 provides a theoretical background for the application of multiparametric LP to generate efficient set $\mathcal{E}$. M. Zeleny discusses this approach in [18] and its practical application is possible for two objective functions. We shall discuss this approach further on.

Observe the other meaning of theorem 5. It states that every efficient solution is an optimal solution for some linear utility function. So, the set of optimal weights associated with particular objective functions is assigned to every efficient solution. While talking about utility functions and its relations with efficient solutions it is interesting to mention P.L.Yu, who proved [18] that for utility function increasing in every argument /the function is defined in criteria space/, the optimal solution of problem /14/ will be an efficient one. Linear utility functions with positive coefficients belong to this class of functions.

4. Compromise solutions in MOLP problems.

4.1. The set $\mathcal{E}$ is usually too large and its analytical formulation too complicated for practical use. That is why we confine our investigation to certain subset $\mathcal{H}$ of $\mathcal{E}$, which we call the set of compromise solutions.

The set $\mathcal{H}$ is very frequently determined as a set of optimal solutions of all/some of single criteria MP problems:

$$\min\left\{ \sum_{i=1}^{t} [h_i(M_i - c_i x)]^k \;\middle|\; x \in \mathcal{X} \right\} \qquad /16/$$

for $k = 1,2,3,\ldots$ and where $h_i > 0$ $/i = 1, \ldots, t/$.

Note that problem /16/ is equivalent to

$$\min\left\{ L_k(x) \mid x \in \mathcal{X} \right\}, \qquad /16'/$$

where $L_k/x/ = \sqrt[k]{\sum_i [h_i(M_i - c_i x)]^k}$ is so called L_k- metric.

Used above term "is equivalent"
means that the sets of optimal solutions /16/ and /16'/ are identical.

It is obvious why we call a point $M = (M_1, \ldots, M_s)$ as an ideal solution of /8/ in criteria space. Usually M does not belong to F(X). By solving /16/ or its equivalent /16'/ for given k, we obtain such a point $\hat{x}^k \in \mathcal{X}$, which image $F/\hat{x}^k/ \in \mathcal{F}$ in criteria space is situated as close as possible to the point $M \in R^t$, with respect to L_k- metric. The optimal solution $\hat{x}^k$ we take as a compromise solution of problem /8/.

If all the functions $c_i x > 0$ for $x \in \mathcal{X}$, we shall use $h_i = 1 : M_i$ $/i = 1, \ldots, t/$ in /16/, otherwise $h_i = 1 : /M_i - m_i/$, where $m_i = \min\{c_i x \mid x \in \mathcal{X}\}$
because then

$$\forall_{1 \leq i \leq t} \ \forall_{x \in \mathcal{X}} \qquad 0 \leq h_i(M_i - c_i x) \leq 1 .$$

P.L.Yu has proved [18] some interesting properties of problem /16/

Theorem 6

When $\mathcal{X}$ is bounded, then the optimal solutions $\hat{x}^k$ of /16/ are efficient for k = 1,2,3,... and $F/\hat{x}^k/$ is the unique optimal solution in the criteria space.

For k = 1 the problem /16/ can be transformed to LP problem:

$$\max\left\{\left(\sum_{i=1}^{t} c_i h_i\right) x \;\middle|\; x \in \mathcal{X}\right\}. \qquad /17/$$

For k = 2 the problem /16/ belongs to the class of convex quadratic programming problems.

For $k \geqslant 2$ all of the problems are NLP problems with convex objective function, what guarantees that there are unique optimal solutions $\hat{x}^k$ in decision space.

4.2. While it is reasonable to motivate the choice of degree k for $L_k/x/$ metric if we take into account $k = 1$ or $k = 2$ and compute final optimal solution of problem /8/ with one of the metrices only, it is not so when we consider $k_1 \neq k_2$ and $k_1, k_2 \geqslant 3$. That is a reason why in practise we use L_1, L_2 and also L_∞ metric to calculate compromise solutions in /8/. The L_∞ metric express limiting tendency of L_k metrices when $k \to \infty$. So

$$L_\infty(x) = \lim_{k\to\infty} L_k(x) = \max_{1\leqslant i\leqslant t} \{h_i(M_i - c_i x)\} \text{ for } x \in \mathcal{X}.$$

The problem /8/ for $k = \infty$ is then

$$\min\Big[\max_{1\leqslant i\leqslant t}\{h_i(M_i - c_i x)\} \mid x \in \mathcal{X}\Big]. \qquad /18/$$

The problem /18/, known as Chebychev problem can be transformed to the following LP problem:

$$z \to \min \qquad /19/$$

s.t.

$$\begin{cases} \mathbb{A}x = \mathbb{b} \\ \mathbb{C}x + \mathbb{h}^{-1}z \geqq \mathbb{M} \\ x \geqslant \mathbb{O}, \end{cases} \qquad /20/$$

Where $\mathbb{h}^{-1} = [h_1^{-1}, \ldots, h_t^{-1}]^T$. It is possible to prove [11] the theorem analogous to theorem 1 for the problem /19/ - /20/.

4.3. Metrices L_1 and L_∞ occupy the extreme positions in a family of L_k metrices. That is why some suggestions concerning compromise solutions deal with L_1 and L_∞ metrices. As a first we propose to consider two approaches, both leading to quadratic programming problem. They are two-stage procedures.

In the first stage we compute solutions $\hat{x}_1$ and $\hat{x}_\infty$, and in the second stage we solve problem, which provides compromise solution. In the first approach it is a problem:

$$\| \mathbb{C}x - \mathbb{M} \|^2 \longrightarrow \min \qquad /21/$$

s.t.

$$\begin{cases} \mathbb{A}x = \mathbb{b} \\ L_1(x) \leq L_1(\hat{x}_\infty) \\ L_\infty(x) \leq L_\infty(\hat{x}_1) \\ x \geq \mathbb{0}. \end{cases} \qquad /22/$$

In [12] it was shown that constraints /22/ can be transformed to linear form:

$$\begin{cases} \bar{\mathbb{A}}x = \bar{\mathbb{b}} \\ \mathbb{C}x \geq g \\ x \geq \mathbb{0}, \end{cases} \qquad /22'/$$

where $g = [g_1,\dots,g_t]^T$ and simultaneously $g_i = \max\{\mathbb{c}_i\hat{x}_1 ; \mathbb{c}_i\hat{x}_\infty\}$. If the first constraint in /22/ includes the requirement $\mathbb{c}_i x \geq d_i$ and also $d_i < g_i$, it is redundant in /22'/. That is why matrix $\mathbb{A}$ is replaced by $\bar{\mathbb{A}}$ and vector $\mathbb{b}$ is replaced by $\bar{\mathbb{b}}$ in /22'/.

In the second approach, one gets compromise solution solving:

$$\| y_1 F_1 + y_2 F_\infty + y_3 \mathbb{M} - \mathbb{M} \|^2 \longrightarrow \min \qquad /23/$$

s.t.

$$\begin{cases} \mathbb{A}x = \mathbb{b} \\ \mathbb{C}x - y_1 \mathbb{F}_1 - y_2 \mathbb{F}_\infty - y_3 \mathbb{M} = \mathbb{0} \\ y_1 + y_2 + y_3 = 1 \end{cases} \qquad /24/$$

$$x \geq \mathbb{0} \quad y_i \geq 0 \quad (i = 1,2,3)$$

where $\mathbb{F}_1 = \mathbb{F}(\hat{x}_1)$, $\mathbb{F}_\infty = \mathbb{F}(\hat{x}_\infty)$

Both approaches define compromise solution as a point from the subset of $\mathcal{X}$, determined by $\hat{x}_1$, and $\hat{x}_\infty$ /differently for each approach/, which has an image F/x/ in criteria space, situated as close as possible in a sense of L_2 metric to ideal solution.

4.4. Frequently , decision maker can not precise definite values of weights attached to particular goals, what is necessary while constructing additive utility function. Also he is not able to provide sufficient information required for the estimation of utility function parameters . On the other hand decision maker's experience helps him to define intervals $\langle \alpha_i ; \beta_i \rangle \subset \langle 0; 1 \rangle$ for particular weights p_i. This kind of additional information is also helpful in utility function construction. Computation of the problem /8/ optimal solution can be replaced by solving:

$$y^T \mathbb{C} x \longrightarrow \max \qquad /25/$$

s.t.

$$\begin{aligned} &[1]^T y = 1 \\ &\alpha \leqslant y \leqslant \beta \\ &\qquad \mathbb{A} x = \mathbb{b} \\ &\qquad x \geqslant 0 \end{aligned} \qquad /26/$$

and $\alpha \geqslant 0$, $\beta < [1]$.

The problem /25/ - /26/ is so called bilinear programming problem, with boundéd solutions set. This gives an opportunity to solve problem /25/ - /26/ by some simplex procedure [3].

5. On some M.Zeleny proposal.

5.1. M.Zeleny proposed [18] to define set $\mathcal{H}$ of problem /8/ as the efficient solutions set of the following bicriterial mathematical programming problem:

$$\min \left\{ \begin{bmatrix} L_1(x) \\ L_\infty(x) \end{bmatrix} \;\middle|\; x \in \mathcal{X} \right\} \qquad /27/$$

Notice that the objective functions in /27/ are the aggregates of all objective functions of problem /8/. Also the problem /27/ is interesting, because they can be replaced according to theorem 5 by the one parametric problem:

$$\min\{(1-\lambda)L_1(x) + \lambda L_\infty(x) \mid x \in \mathcal{X}\} \text{ for } \lambda \in \langle 0;1 \rangle .$$

For $\lambda = 0$ the above problem is equivalent to /16/ with k = 1, and for $\lambda = 1$ it is equivalent to /16/ with k = ∞ or in other words to /18/. It is easy to show that this problem can be transformed to the one-parametric LP problem:

$$(\lambda - 1)\left(\sum_{i=1}^{t} h_i \mathbb{C}_i\right) x - \lambda z \longrightarrow \max \qquad /28/$$

s.t.

$$\begin{cases} \mathbb{A} x = \mathbb{b} \\ \mathbb{C} x + \mathbb{h}^{-1} z = \mathbb{M} \quad \text{for } \lambda \in \langle 0;1 \rangle . \\ x \geqslant \mathbb{O}. \end{cases} \qquad /29/$$

By solving /28/ - /29/ we get the following division:

$$\langle 0;1 \rangle = \bigcup_{j=1}^{p} \langle \lambda_{j-1} ; \lambda_j \rangle \qquad /30/$$

where $\lambda_0 = 0$, $\lambda_p = 1$ and a given optimal solution $\bar{\bar{x}}_j$ is associated with the interval $\langle \lambda_{j-1}, \lambda_j \rangle$ for j = 1,...,p and simultaneously $\bar{\bar{x}}_1 = \hat{x}_1$ and $\bar{\bar{x}}_p = \hat{x}_\infty$. In fact division /30/ defines the sequence of $r \geqslant p$ different basic optimal solutions of /28/-/29/:

$$\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_r$$

because more than two optimal solutions may be associated with some critical values λ_j. Most of the solutions $\tilde{x}_j$ are adjacent to its neighbours and obviously $\tilde{x}_1 = \hat{x}_1$, and $\tilde{x}_r = \hat{x}_\infty$

According to theorem 5 we know that all r solutions $\tilde{x}_j$ are efficient basic solutions of /28/ - /29/. With respect to our earlier assumptions we take:

$$\mathcal{K} = \mathcal{E}_{L_1, L_\infty} = \bigcup_{j=1}^{r-1} \{ x \mid (1-\gamma_j)\tilde{x}_j + \gamma_j \tilde{x}_{j+1} ;\ 0 \leq \gamma_j \leq 1 \} \qquad /31/$$

Naturally $\mathcal{K} \subset \mathcal{X}$ and $\mathcal{K} = \mathcal{E}_{L_1, L_\infty}$ means that every $x \in \mathcal{K}$ is efficient with respect to the minimization of L_1 and L_∞ functions, but we want to know if it is efficient with respect to maximization of all $c_i x$, so if $\mathcal{K} \subset \mathcal{E}$.

In [13] one proves that

Theorem 7

The following relation holds for the set given by /31/:

$$\mathcal{E}_{L_1, L_\infty} \subset \mathcal{E}.$$

Therefore formula /31/ defines the set of compromise solutions in problem /8/.

5.2. The set $\mathcal{K}$ defined by /31/ can be too large for the decision maker in his search for final solution of problem /8/. Then two approaches are possible.

The first approach. For each $\tilde{x}_j$ we determine the set of weights in linear utility functions which have $\tilde{x}_j$ as their optimal solution. So it is

$$\mathcal{P}_j = \{ p \mid p^T C \tilde{x}_j = \max\{ p^T C x \mid x \in \mathcal{X} \},\ p \geq 0,\ [1]^T p = 1 \}$$

$$/j = 1,2,\ldots,r/ .$$

The final solution can be taken by the decision maker on the basis of his evaluation of goodness of weights attached to particular goals given by $p \in \mathcal{P}_j$

<u>The second approach.</u> It is more formal approach than the first one. We take the set $\mathcal{Q}$, which is the image of $\mathcal{H}$ in the space $L_1 X L_\infty$. If we introduce the notations:

$$u_i = L_1/\tilde{x}_i/ \quad , \quad v_i = L_\infty/\tilde{x}_i/ \quad \text{for } i = 1, \ldots, r$$

we obtain r points

$$y_i = \begin{bmatrix} u_i \\ v_i \end{bmatrix} \quad L_1 X L_\infty \qquad /i = 1, \ldots, r/$$

defining set $\mathcal{Q}$ in the following way

$$Q = \bigcup_{i=1}^{r-1} \{ y \mid y = (1-\gamma_i) y_i + \gamma_i y_{i+1} \, , \; 0 \leq \gamma_i \leq 1 \}. \qquad /32/$$

Next we determine a point $y_0 = [u_0, v_0]^T$ taking either $u_0' = L_1 /x_M/$, $v_0' = L_\infty /x_M/$, where x_M is such, that $F/x_M/ = M$ or $u_0'' = L_1/\hat{x}_1/$, $v_0'' = L_\infty/\hat{x}_\infty/$.

The vector y_0' represents the image of ideal solutions of /8/ in $L_1 X L_\infty$ space.

The vector $y_0'' \neq y_0'$ and y_0'' represents the ideal solution of /27/ in criteria space $L_1 X L_\infty$.

The solution vector $\hat{x}$ will be taken as a such point of $\mathcal{H}$, that $\hat{y} = [L_1/\hat{x}/, L_\infty/\hat{x}/]^T \in \mathcal{Q}$ is the nearest point from y_0 among all elements of $\mathcal{Q}$, when the distance is expressed by Euclidian metric, i.e.

$$\|\hat{y} - y_0\|^2 = \min_{1 \leq i \leq r-1} \{ \min_{0 \leq \gamma_i \leq 1} \|(1-\gamma_i) y_i + \gamma_i y_{i+1} - y_0\|^2 \}. \qquad /33/$$

5.3. We do not need to use any specific MP procedure to find $\hat{x}$ by /33/. The problem /33/ may be put into the following form by very simple algebraic operations:

$$\min_{1 \leq i \leq r-1} \{ \min_{0 \leq \gamma_i \leq 1} \varphi_i(\gamma_i) \} \qquad /33'/$$

where $\varphi_i(\gamma_i) = a_i \gamma^2 + b_i \gamma_i + c_i$ and

$$a_i = A_i^2 + B_i^2 \qquad\qquad A_i = u_{i+1} - u_i$$

$$b_i = -2/A_i C_{i+1} + B_i D_{i+1}/ \quad \text{and} \quad B_i = v_{i+1} - v_i$$

$$c_i = C_i^2 + D_i^2 \qquad\qquad C_i = u_i - u_o$$

$$D_i = v_i - v_o$$

$\varphi_i/\gamma_i/$ is a square function with coefficients $a_i > 0$ for $i = 1,2,\ldots, r-1$. So φ_i reaches its global minimum at

$$\gamma_i^m = -b_i : 2a_i \quad (i=1,\ldots,r-1).$$

Let $\gamma_i^o \in \langle 0;1 \rangle$ denote such number, that

$$\varphi_i(\gamma_i^o) = \min\{\varphi_i(\gamma_i) \mid 0 \leq \gamma_i \leq 1\}.$$

Therefore

$$\gamma_i^o = \begin{cases} 0 & \text{when} \quad \gamma_i^m \leq 0 \\ \gamma_i^m & \text{when} \quad 0 < \gamma_i^m < 1 \\ 1 & \text{when} \quad \gamma_i^m \geq 1 \end{cases} \qquad \text{for } i=1,\ldots,r-1 \qquad /34/$$

Having caltulated $\varphi_i^o = \varphi_i(\gamma_i^o)$ for $i = 1,\ldots r-1$ one can replace problem /33/ by a new one:

$$\|\hat{y} - y_o\|^2 = \hat{\varphi}_v = \min\{\varphi_i^o \mid i = 1,2,\ldots,r-1\}. \qquad /35/$$

Observe the shape of set Q and its special location with respect to the point y_o, what is shown on fig.4. These properties of the set Q follow from its definition, y_o determination and theorem 4. It allows us to obtain solution $\hat{x}$ without necessity of calculating all φ_i^o or even A_i, B_i, C_i, D_i at the start of problem /35/ solving. These quantities may be constructed gradually, and procefure is terminated when one reaches $\hat{\varphi}_v$.

More detailed description of this procedure is given in the paper [8].

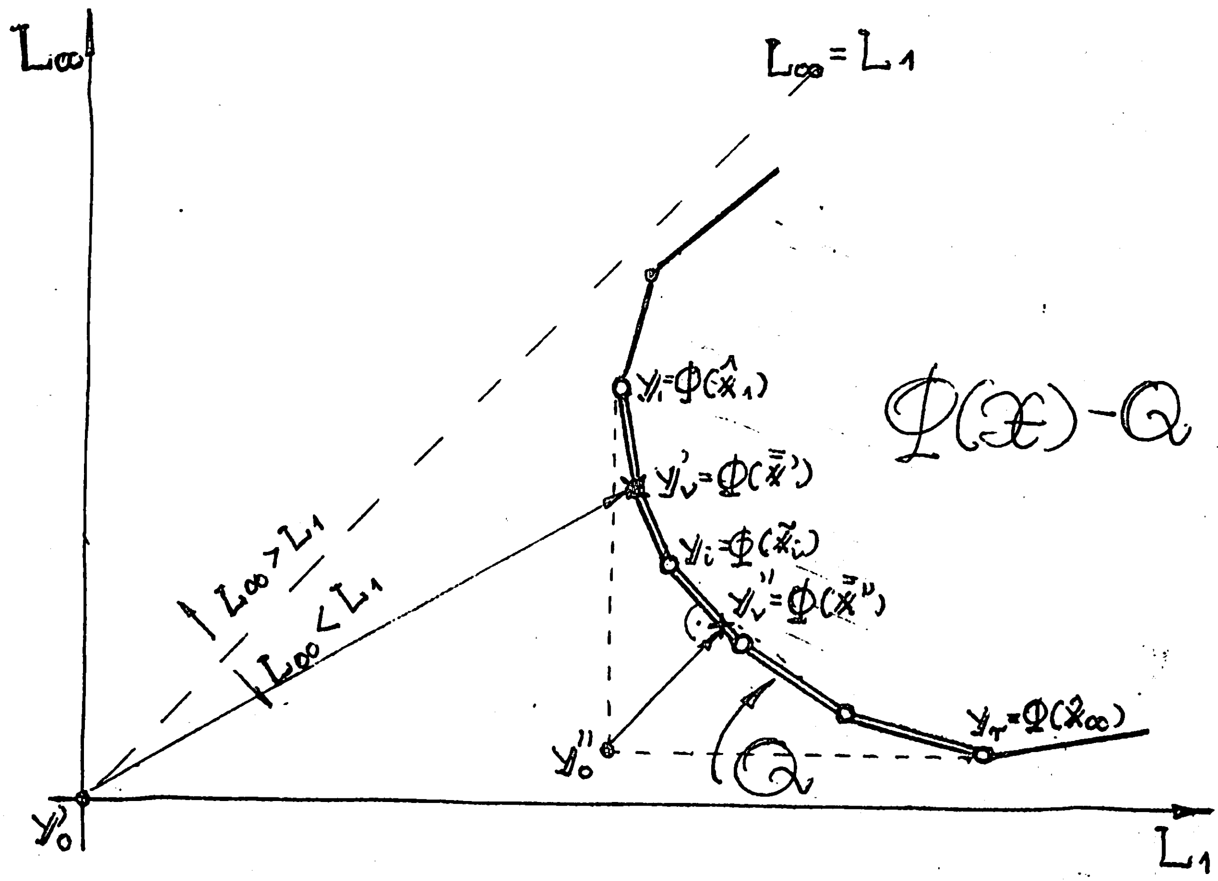

Figure 4 *)

*) The set Q must be located below a straight line $L_\infty = L_1$, because it is easy to prove that $L_\infty(x) < L_1(x)$ for each $x \in \mathfrak{X}$.

Having determined $\hat{\varphi}_v$ we start to look for the solution $\hat{x}$. This depends on one of the two situations, which conducted to $\hat{\varphi}_v$.

1° $\hat{\varphi}_v$ is a unique solution of /35/, then

$$\bar{\bar{x}} = /1 - \delta_v^o/ \tilde{x}_v + \delta_v^o x_{v+1}$$

where δ_v^o is computed according to /34/ and

$$0 < \delta_v^o < 1 \qquad \text{if} \quad 1 < v < r - 1$$

$$0 \leqslant \delta_v^o < 1 \qquad \text{if} \quad v = 1$$

$$0 < \delta_v^o \leqslant 1 \qquad \text{if} \quad v = r-1 .$$

2° There are two optimal values $\varphi_v^o = \varphi_{v+1}^o$, then $\delta_v^o = 1$, $\delta_{v+1}^o = 0$ and a final solution is:

$$\bar{\bar{x}} = \tilde{x}_{v+1} .$$

5.4. Finally it should be stressed that we did not mention about interactive programming, while discussing the procedures dealing with compromise solutions determination. The interactive methods are becoming more popular nowadays, because of their "interactive" character between the researcher and decision maker during computation phase. It seems that very good review and some interesting proposals dealing with the application of interactive programming to compromise solutions determination, can be found in paper [20] written by I. Wallenius.

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