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ON THE REDUCTION OF LP PROBLEM

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BSTRACT

P problems reduction prior to optimal solutions finding and during computational procedure is discussed in this paper. So, the notions of redundant constraints /§§ 2,3/, conditionally redundant constraints and conditionally redundant variables /§4/ are introduced.

Redundant constraints are defined as these not changing feasible solutions set $\mathcal{X}$ when removed from its definition. The constraints of the second type are those, which are not active in optimal solutions set $\mathcal{X}_0$ of LP problem.

By conditionally redundant variables we understand these having zero value in optimal solution. Removal of the redundant variables and constraints from further considerations does not effect optimal solution choice, but frequently leads to a significant reduction of LP problem size.

In §§ 2,3, sufficient conditions for redundant constraints and in §4 for conditionally redundant constraints and variables detection are discussed.

Two methods of LP problems reduction presented in [7], [9] are analysed as well.

1. INTRODUCTION [1/]

Speaking about the LP problems reduction we have on mind the diminution of problem size prior to numerical procedure leading to optimal solution, if one exists. By the size of LP problem we understand the number of significant constrai /i.e. all constraints different from nonnegativity conditi defining feasible solutions set $\mathfrak{X}$ and the number of deci variables in this problem.

It is well known that too big number of the constraints def set $\mathfrak{X}$ is inconvenient when looking for optimal solutions because it lengthens computing time and effects accuracy o the results. Given an objective function and a type of optimization /maximization or minimization/, only some cons traints defining the set $\mathfrak{X}$, define also optimal solutio set $\mathfrak{X}_0$ $(\mathfrak{X}_0 \subset \mathfrak{X})$. Neglecting LP constraints /or some of th not defining $\mathfrak{X}_0$ set can reasonably shorten the computati work leading to $\mathfrak{X}_0$ determination.

The special place among all constraints defining set $\mathfrak{X}$ occupied by these which do not influence $\mathfrak{X}$, while removed

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1/ This report is, in a large part, based on papers presen by the author and others at SGPiS seminar on "Numerical thods of large scale optimization models", chaired by p W.Grabowski. The seminar was sponsored by the Institut Planning at the Council of Ministers, and its results w published in /6/. Some unpublished theorems and notions troduced by dr W. Dubnicki are used in this report as w the problem. Such constraints we call redundant. We shall introduce the notion of conditionally redundant constraints i.e. the constraints of $\mathcal{X}$ not determining the set $\mathcal{X}_o$.

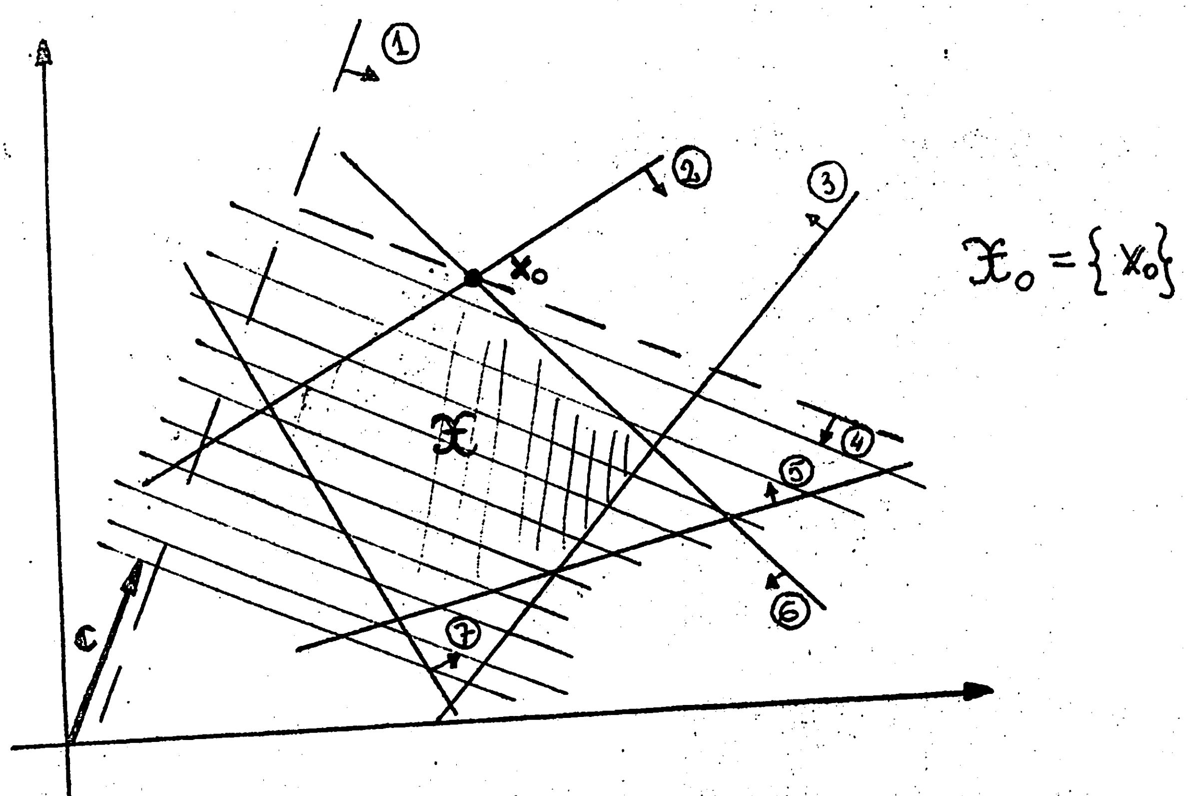

Figure 1

Fig. 1 presents set $\mathcal{X}$ of some LP problem, where objective function is maximized. Constraints of this problem, numbered 1 and 4 are redundant. Conditionally redundant are constraints numbered 1, 3, 5, 7.

Let's assume that LP problem will be solved by some specific type of simplex method. In this method nonnegativity conditions are automatically satisfied, so they are not introduced into computer storage, and they do not increase the size of LP problem.

Considering redundant or conditionally redundant constraints, we shall take into account all but nonnegativity conditions. In some cases nonnegativity conditions can play important role in the determination of $\mathcal{X}$ or $\mathcal{X}_0$, therefore it is reflected in the notion of redundant or conditionally redundant variable.

2. Redundant inequalities

A. Our further considerations will deal with the following LP problem, formulated here in canonical form:

$$\mathbf{c}\mathbf{x} \longrightarrow \max \qquad /1/$$

s.t.

$$\mathbf{a}_i\mathbf{x} \leqslant b_i \quad (i=1,\dots,m) \qquad /2/$$

$$\mathbf{x} \geqslant \mathbf{0}, \qquad /3/$$

where $\mathbf{x}, \mathbf{c}, \mathbf{a}_i$ are vectors in R^n.

Canonical formulation of LP problem is equivalent to standard and mixed forms, what results
from the equivalence of transformations:

$$\mathbf{a}_i\mathbf{x} \leq b_i \equiv \begin{pmatrix} \mathbf{a}_i\mathbf{x} + x_{n+i} = b_i \\ x_{n+i} \geqslant 0 \end{pmatrix} \qquad /4a/$$

and

$$\mathbf{a}_i\mathbf{x} = b_i \equiv \begin{pmatrix} \mathbf{a}_i\mathbf{x} \leqslant b_i \\ -\mathbf{a}_i\mathbf{x} \leqslant -b_i \end{pmatrix} \qquad /4b/$$

Writing /1/ - /3/ form, we want to stress that in initial formulation of LP problem, all constraints different from /3/ were written as inequalities. It will be shown further, that differ rules govern redundancy determination in case of inequalities, than in case of equations. The following notation results from /1/ - /3/ formulation:

$R_+^n = \{x \mid x \in R^n, x \geq 0\}$, $J = \{1, 2, \dots, m\}$, $J_{/k} = J - \{k\}$,

$\mathcal{X} = \{x \mid x \in R_+^n, a_i x \leq b_i \ (i \in J)\}$,

$\mathcal{X}_{/k} = \{x \mid x \in R_+^n, a_i x \leq b_i \ (i \in J_{/k})\}$,

$S_k = \{x \mid x \in R_+^n, a_k x \leq b_k\}$ and $k \in J$,

$T_k = \{x \mid x \in R_+^n, x \in \mathcal{X}_{/k} - S_k\}$,

From the above definitions it follows immediately:

$$\mathcal{X} \subset \mathcal{X}_{/k} \text{ and } \mathcal{X} \subset S_k. \qquad /5/$$

<u>Definition 1</u>

Constraint "k" of system /2/ is redundant in $\mathcal{X}$ iff

$$\mathcal{X}_{/k} \subset S_k.$$

From the def.1 and relation /5/ the following holds:

<u>Theorem 1</u>

The following relations:

/i/ $\mathcal{X}_{/k} \subset S_k$,

/ii/ $\mathcal{X}_{/k} \subset \mathcal{X}$,

/iii/ $\mathcal{X}_{/k} = \mathcal{X}$,

can be used in the redundancy definition of "k" constraint, because they are mutually equivalent.

In the definition of the set $\mathcal{X}$, it is convenient to introduce the notion opposite to redundant constraint, i.e. essential constraint. We assume that the set $\mathcal{X}$ will be changed by the removal of a essential constraint.

Definition 2

The constraint "k" in system /2/ is essential in $\mathcal{X}$, iff

$$\mathcal{X}_{/k} \not\subset S_k .$$

From the definition 1, theorem 1 and definition 2 we get:

Theorem 2

The following relations:

/i/ $\mathcal{X}_{/k} \not\subset S_k$,

/ii/ $\mathcal{X}_{/k} \neq \mathcal{X}$,

/iii/ $\mathcal{J}_k \neq \emptyset$,

can be used in the number "k" essential constraint definition, because they are mutually equivalent.

From the numerical point of view, the following theorems are more important than theorem 1 and 2 [2/].

Theorem 3.

"k" constraint in system /2/ is essential, iff

$$\mathcal{J}_k(t) = \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \middle| \begin{array}{ll} a_i x \leqslant t b_i & i \in J_{/k} \\ a_k x > t b_k & \\ x \geqslant 0, \; t \geqslant 0 & \end{array} \right\} \neq \emptyset$$

2/ Theorem 3, as well as necessary condition in theorem 4, were formulated and proved by W.Dubnicki at SGPiS seminar and they will be published. Proof of the sufficient condition in theorem 4 is given by the author, as a modification of this condition, as presented in /4/.

Theorem 4

The constraint "k" in system /2/ is redundant in $\mathcal{X}$, iff:

$$V_{/k} = \left\{ v = [v_i]_{i \in J_{/k}} \;\middle|\; \begin{array}{l} a_k \leq \sum_{i \in J_{/k}} v_i a_i \\ b_k \geq \sum_{i \in J_{/k}} v_i b_i \\ v \geq 0 \end{array} \right\} \neq \emptyset$$

Having in mind the numerical importance of the sufficient conditions in theorem 4, here we present its proof.

Let's assume that $V_{/k} \neq \emptyset$. For any $x \in \mathcal{X}_{/k}$ and $v \in V_{/k}$ we get

1° $a_k x \leq \left(\sum_{i \in J_{/k}} v_i a_i \right) x$

2° $\sum_{i \in J_{/k}} v_i (a_i x) \leq \sum_{i \in J_{/k}} v_i b_i$

3° $\sum_{i \in J_{/k}} v_i b_i \leq b_k$

From 1°, 2°, 3° we have $\left(x \in \mathcal{X}_{/k} \Rightarrow x \in S_k \right) \Rightarrow \mathcal{X}_{/k} \subset S_k$)

so the constraint "k" is redundant in $\mathcal{X}$. The necessary and sufficient condition in theorem 4 describes a rule of practical determination constraint "k" redundancy in $\mathcal{X}$.

Corollary 1

If the following system of inequalities has a solution

$$\begin{cases} \sum_{i \in J_{/k}} a_i v_i \geq a_k \iff \left(\sum_{i \in J_{/k}} a_{ij} v_i \geq a_{kj} \quad (j=1,\dots,n) \right) \\ \sum_{i \in J_{/k}} b_i v_i \leq b_k \\ \qquad v_i \geq 0 \quad (i \in J_{/k}) \end{cases} \qquad /6/$$

then, constraint "k" is redundant in $\mathcal{X}$.

Solution of the above system of linear inequalities for v_i can be reached by solving of the LP problem

$$z \longrightarrow \max \qquad /7/$$

s.t.

$$\begin{cases} \sum\limits_{i \in J_{/k}} a_i v_i \geqslant a_k \\ \sum\limits_{i \in J_{/k}} b_i v_i + z = b_k \end{cases} \qquad /8/$$

$$v_i \geqslant 0 \quad (i \in J_{/k}).$$

If in the optimal solution $[\bar{v}^T \bar{z}]$ of /7/ - /8/ we have $\bar{z} \geqslant 0$, so $\bar{v} \in V_{/k}$, what indicates the redundancy of "k" constraint. The problem /7/ - /8/ is solved until basic solution satisfying

$$[v^T, z] \geqslant [0^T, 0]$$

is met.

If in the optimal solution of /7/ - /8/ $\bar{z} < 0$, than according to a necessary condition of theorem 4, the constraint "k" is not redundant.

B. The set $V_{/k}$ is defined by n + 1 inequalities. Taking into account the system /2/, generating the problem /7/-/8/, usually it is true that $n > m$. Hence, m constraints in system /2/ created numerical difficulties, there is no usefulness in applying problem /7/ - /8/.

Fortunately the situation is not so bad. Considering a solution of system /6/ it is possible to assume that some $v_i = 0$, what practically means a possibility to reduce the number of inequalities in system /2/, searched for a redundant constraint.

It is easy to prove,

<u>Theorem 5</u>

If a linear inequality is redundant in relation to a given set of m linear inequalities, it is also redundant in relation to this set expanded by p additional inequalities.

Utilizing theorem 5 we present some modifications of corollary 1, resulting in an efficient numerical procedure of a redundancy determination, applied in relation to the whole or partial system /2/. Let $\mathcal{J}(k) \subset \mathcal{J}_{/k}$, and r be a quantity of subset $\mathcal{J}(k)$ ($r = \operatorname{card} \mathcal{J}(k)$).

Theorem 4 - I

If for some $\mathcal{J}(k)$ the following inequalities hold

$$\begin{cases} a_{kj} \leq \frac{1}{r} \sum_{i \in J(k)} a_{ij} \quad (j=1,\dots,n) \\ b_k \geq \frac{1}{r} \sum_{i \in J(k)} b_i , \end{cases} \qquad /9/$$

then constraint "k" from system /2/ is redundant in $\mathcal{X}$.

To prove the theorem 4-I it is enough to see, that it is a special case of corollary 1, where $v \in V_{/k}$ is a vector with coordinates

$$v_i = \begin{cases} \frac{1}{r} & \text{for } i \in J(k) \\ 0 & \text{for } i \in J_{/k} - J(k). \end{cases}$$

Theorem 4-II

If for some $\mathcal{J}(k)$ the following inequalities hold

$$\begin{cases} a_{kj} \leq \sum_{i \in J(k)} a_{ij} \quad (j=1,\dots,n) \\ b_k \geq \sum_{i \in J(k)} b_i . \end{cases} \qquad /10/$$

then constraint "k" from system /2/ is redundant in $\mathcal{X}$.

Justification of theorem 4-II is analogous to theorem 4-I.

As the components of vector v, one takes

$$v_i = \begin{cases} 1 & \text{for } i \in J(k) \\ 0 & \text{for } i \in J_{/k} - J(k) \end{cases}$$

It is easy to check inequalities /9/ and /10/ and they can be used for large systems of inequalities. For theorem 4-I we use arithmetical means. It is convenient if coefficients for particular variables and RHS* in constraints "i" $\in J(k)$ are of the same order as those in constraint "k". Application of theorem 4-II is proposed when significant disproportions between constraint "k" coefficients and reference constraints $i \in J(k)$ are observed.

The significant property of the set $V_{/k}$ definition is a sign discretion of coefficients appearing in its inequalities, i.e. in inequalities /9/ and /10/. For positive values b_i of RHS, system /2/ can be written in a corresponding form

$$\bar{a}_i x \leq 1 \qquad (i \in J) \qquad /2'/$$

$$x \geq 0, \qquad /3'/$$

where $\bar{a}_{ij} = a_{ij} : b_i$.

Application of theorem 4-I to the system /2'/, /3'/ results in change of an inequality in /9/ to identity 1=1. So we get:

<u>Theorem 4-III</u>

If in the set $\mathcal{X}$ defined by /2'/, /3'/ for some $J(k)$ it holds:

$$\bar{a}_{kj} \leq \frac{1}{T} \sum_{i \in J(k)} \bar{a}_{ij} \qquad (j = 1, \dots, n) \qquad /11/$$

so, the constraint "k" is redundant in $\mathcal{X}$.

If the set $J(k)$ consists of one element for given $k \in J$, then theorem 4-III can be reduced to previously proved by [redacted] W.Grabowski [2] sufficient condition of inequality "k" deletion from system /2/-/3/, because of its redundancy in reference to inequalities $i \in J(k)$, when $b_i > 0$, $b_k > 0$.

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* RHS - right hand side

This condition is formed by the system of inequalities:

$$\frac{a_{kj}}{b_k} \leq \frac{a_{ij}}{b_i} \quad (j=1,\dots,n) \qquad /11'/$$

or

$$\bar{a}_{kj} \leq \bar{a}_{ij} \quad (j=1,\dots,n),$$

what equals formula /11/ for r= 1.

Theorem 4-III can not be applied to delete the equations formed from inequalities, by introduction of nonnegative slack variables. Let inequalities "i" and "k" from system /2/ satisfy conditions /11'/. The corresponding equations are:

$$a_i x + x_{n+i} = b_i \; (b_i > 0),$$
$$a_k x + x_{n+k} = b_k \; (b_k > 0).$$

These equations do not satisfy conditions /11'/, though

$$0 \equiv \frac{a_{k,n+i}}{b_k} < \frac{a_{i,n+i}}{b_i} \equiv \frac{1}{b_i},$$

however

$$\frac{1}{b_k} \equiv \frac{a_{k,n+k}}{b_k} > \frac{a_{i,n+k}}{b_i} \equiv 0.$$

C. We shall discuss two other approaches when theorem 4-III, applied to the two inequalities in system /2/, does not lead to determination of the constraint "k" redundancy. Note, that theorems 4-I, 4-II and 4-III are the simplifications of theorem 4. They form sufficient, but not necessary conditions for constraint "k" redundancy.

First case. We consider two inequalities from system /2/, numbered "k" and "i" respectively. Their coefficients do not satisfy condition /11'/, because

$$\bar{a}_{kj} \leq \bar{a}_{ij} \quad \begin{cases} j=1,\dots,n \\ j \neq p \end{cases} \qquad /a/$$

while

$$\bar{a}_{kp} > \bar{a}_{ip}. \qquad /b/$$

Therefore, one can not say that the constraint "k" is redundant. Let us try to apply corollary 1 to these inequalities, i.e. we check if exists v satisfying inequalities:

$$\begin{cases} \bar{a}_{ij} v \geqslant \bar{a}_{kj} & j=1,\dots,n \; j \neq p & /c/ \\ \bar{a}_{ip} v \geqslant \bar{a}_{kp} & & /d/ \\ 0 \leq v \leq 1. & & /e/ \end{cases}$$

Because of the third inequality, the above system has a solution, if the following possibilities are excluded:

1° $\bar{a}_{ip} \geqslant 0$,

2° $\bar{a}_{ip} \cdot \bar{a}_{kp} \leq 0$,

3° $\bar{a}_{kj} = \bar{a}_{ij} > 0$.

From inequality /a/, (c), (e), we get

$$v \geqslant \max\left\{0;\ \frac{\bar{a}_{kj}}{\bar{a}_{ij}} \,\middle|\, \bar{a}_{ij} > 0\right\} \equiv L. \qquad /12/$$

Whereas from inequalities /b/, 1° and 2° we get

$$|\bar{a}_{ip}| > |\bar{a}_{ik}| \geqslant 0$$

From /b/, /d/ and /e/ it holds

$$v \leq \frac{\bar{a}_{kp}}{\bar{a}_{ip}} \equiv \mathcal{U}. \qquad /13/$$

Finally, if the inequality

$$L \leq \mathcal{U}$$

is true, then the constraint "k" is redundant, because by the terms /12/ and /13/ system /c/ - /e/ has a solution.

Suppose now, inequality /b/ holds for more than one p. The set of such p we denote by $\mathcal{P}$. By corollary 1 the constraint "k" is redundant, when the system of inequalities

$$\begin{array}{lll} \bar{a}_{ij} v \geqslant \bar{a}_{kj} & j=1,\dots,n\,;\ j \notin \mathcal{P} & /c'/ \\ \bar{a}_{ip} v \geqslant \bar{a}_{kp} & p \in \mathcal{P} & /d'/ \\ 0 \leq v \leq 1 & & /e'/ \end{array}$$

has a solution. This solution exists if the following three possibilities are excluded:

1^o $\bar{a}_{ip} \geqslant 0 \quad p \in \mathcal{P}$,

2^o $\bar{a}_{ip} \cdot \bar{a}_{kp} \leqslant 0 \quad p \in \mathcal{P}$

3^o $\bar{a}_{kj} = \bar{a}_{ij} > 0$.

In parallel way as previously, we get

$$v \geqslant \max\left\{0;\ \frac{\bar{a}_{kj}}{\bar{a}_{ij}} \,\middle|\, \bar{a}_{ij} > 0\right\} \equiv L$$

and

$$v \leqslant \min\left\{\frac{\bar{a}_{kp}}{\bar{a}_{ip}} \,\middle|\, p \in \mathcal{P}\right\} \equiv \mathcal{U}.$$

Satisfaction of inequality $L \leqslant U$ is sufficient for the constraint "k" to be redundant.

<u>Second case</u>. In the system /2'/ we analyse by double checking of /11'/ the redundancy of constraint "k" in reference to the two other constraints "i" and "h". As a result we get

$$\bar{a}_{kj} \leqslant \bar{a}_{ij} \qquad \begin{cases} j = 1, \ldots, n \\ j \notin \mathcal{P} \end{cases}$$

$$\bar{a}_{kj} \leqslant \bar{a}_{hj} \qquad \begin{cases} j = 1, \ldots, n \\ j \notin Q \end{cases}$$

$\bar{a}_{kp} > \bar{a}_{ip}$ $p \in \mathcal{P}$ and $\bar{a}_{kq} > \bar{a}_{iq}$, $q \in Q$,

where $\mathcal{P}$ and Q are the sets of indices attached to relations not satisfying /11'/, while the constraint "k" is succesively compared to the constraint "i" and later to the constraint "h". The above presented results do not enable to define the constraint "k" as redundant. Therefore we apply corollary 1 to analyse the constraints "k", "h" and "i". So we state that the constraint "k" is redundant if the following LP problem has a solution[3].

3/ It is enough to have any feasible solution of this problem so its consistency is sufficient.

$$v_i + v_h \longrightarrow \max$$

s.t.

$$\bar{a}_{ij} v_i + \bar{a}_{hj} v_h \geqslant \bar{a}_{kj} \quad /j=1,\ldots\ldots,n/$$

$$v_i + v_h \leqslant 1 \qquad v_i \geqslant 0, \quad v_h \geqslant 0.$$

There again, as in the first case it is possible to give conditions of problem inconsistency. For large n it is prefer red to replace this. problem solving b its dual problem , as in fact we need only to check its consistency. The du- al problem is:

$$y_0 + \sum_{j=1}^{n} \bar{a}_{kj} y_j \longrightarrow \min$$

s.t.

$$y_0 + \sum_{j=1}^{n} \bar{a}_{ij} y_j \geqslant 1$$

$$y_0 + \sum_{j=1}^{n} \bar{a}_{hj} y_j \geqslant 1$$

$$y_0 \geqslant 0, \quad y_j \leqslant 0 \quad (j=1,\ldots,n).$$

This is a two constraints problem, so it is easy to solve. The existance of optimal solution to dual problem indicates the redundancy of the constraint "k" in the set $\mathcal{X}$ definition.

§3. Redundant equations.

A. The theory of redundant equations in LP problem is more complicated than the theory of redundant inequalities. It results from various reasons. The condition expressed as an equation is more restrictive than inequality expression.

It is seldom in LP problem to have more equations than variables among conditions describing set $\mathcal{X}$.
This alone points a redundancy of some equations. It is known that conditions initially formulated as equations are usually essential constraints, when conditions formed as equations after introduction of slack variables can be deleted as redundant by procedures dealing with inequalities.

Finally, we do not have so efficient elimination algorithms for redundant equation as it is the case for redundant inequalities.

Standard form of LP problem is a starting point for further considerations

$$\mathbf{c}\mathbf{x} \longrightarrow \max \tag{14}$$

s.t.

$$\mathbf{a}_i \mathbf{x} = b_i \quad (i=1,\dots,m), \tag{15}$$

$$\mathbf{x} \geqslant \mathbf{0}. \tag{16}$$

Linear equations are the specific case of the linear inequalities. So we assume:

1° for the problem /14/ - /16/ we define sets $\mathcal{I}$, $\mathcal{I}_{/k}$, Γ^n, $\mathcal{X}$, $\mathcal{X}_{/k}$, $\mathcal{S}_k$ and $\mathcal{T}_k$ similarly as for the problem /1/ - /3/ /§2, part A/, replacing inequalities of the type $\mathbf{a}_i\mathbf{x} \leq b_i$ by equations $\mathbf{a}_i\mathbf{x} = b_i$. For newly defined sets $\mathcal{X}$, $\mathcal{X}_{/k}$, $\mathcal{S}_k$ relations /5/ also hold, because they form relationships necessarily satisfied by the conjunctions of particular sets.

2° - definitions of redundant and essential constraint in feasible solutions set $\mathcal{X}$ are the same for equations as for inequalities. It means that definitions 1 and 2 are valid

From the above assumptions it is obvious that theorems 1 and 2 in relation to problem /14/ - /16/ are valid.

B. Let's start the description of numerical procedures of redundancy determination in system /15/. Denote by $\mathbb{A}$ variables coefficients matrix in system /15/. The rows of matrix $\mathbb{A}$ are formed by vectors a_i from this system.

<u>Theorem 6</u>

The equation "k" of system /15/ is redundant in $\mathcal{X}$, if

$$\mathcal{W} = \{ w \in R^m \mid w \neq 0,\ w^T[\mathbb{A}\ b] = [0, 0] \} \neq \emptyset \qquad /17/$$

and simultaneously $w_k \neq 0$.

From the assumption of theorem 6 it follows that the equation "k" is a linear combination of the remaining equations of /15/, i.e.

$$a_k = \sum_{i \in J_{/k}} t_i a_i \quad , \quad b_k = \sum_{i \in J_{/k}} t_i b_i ,$$

where $t_i = w_i : w_k$ and $w \in \mathcal{W}$.

Therefore we have for any $x \in \mathcal{X}_{/k}$

$$\left[\left(\sum_{i \in J_{/k}} t_i a_i x = \sum_{i \in J_{/k}} t_i b_i \right) \Rightarrow a_k x = b_k \right] \Rightarrow \mathcal{X}_{/k} \subset S_k .$$

So the equation "k" is redundant in $\mathcal{X}$, what completes the proof of theorem 6.

Theorem 6 is equivalent to sufficient condition of theorem 4. Its practical applicability is small, though it states sufficient condition of redundancy detection. Creation of the set $\mathcal{W}$ even for a given subset $\mathfrak{J}(k)$, if done for the big number of equations, should be time-consuming.

It is worthwhile to mention that the set $\mathcal{W}$ exploration can be contracted to a specific mathematical programming problem solving, corresponding[4] to the problem /7/ - /8/. It means to the problem:

$$g(u) = \sum_{i \in J} |u_i| \longrightarrow \max \qquad /18/$$

s.t.

$$\mathcal{U} \sim \begin{cases} u^T \cdot A_j = 0 & (j = 1, \dots, n) \\ u^T b = 0, & \end{cases} \qquad /19/$$

where A_j is j-th column of matrix A .
From /17/ and /19/ we have $\mathcal{U} - \mathcal{W} = \{0\}$ and also $\mathcal{U} \neq \emptyset$, because $0 \in \mathcal{U}$. Function $g(u) \geqslant 0$ for $u \in \mathcal{U}$. So if $\bar{u}$ denotes optimal solution of the problem /18/-/19/, than $\mathcal{W} \neq \emptyset$ iff $g(\bar{u}) > 0$ or $\bar{u} \neq 0$. The vector in demand is $v = \bar{u}$. We can stop to solve[5] the problem /19/ - /20/ when the first feasible basic solution $u \neq 0$ is obtained. If the problem /18/-/19/ is used for "k" equation redundancy testing in system /19/, it is convenient to take $g(u) = |u_k|$ as the objective function.
There exists another approach to equations redundancy determination in system /15/. It consists of two phase simplex method used to solve the problem /14/ - /16/.

4/ Observe the resemblance of conditions describing sets $\mathcal{V}_k$ and $\mathcal{W}$.

5/ We discuss problem /18/ - /19/ solving in the appendix A.

The system /15/ can be written in a block form, if it has equations being linear combinations of the other equations:

$$\begin{bmatrix} \mathbb{A}^{(1)} \\ \mathbb{A}^{(2)} \end{bmatrix} \mathbb{x} = \begin{bmatrix} \mathbb{b}^{(1)} \\ \mathbb{b}^{(2)} \end{bmatrix} , \quad \mathbb{x} \geqslant \mathbb{0}.$$

The first block is formed by linear independent equations and the second block - by linear dependant. In such a case, phase I of simplex approach is terminated, with well known relations:

$$[1]^T \mathbb{s}_B + [1]^T \mathbb{s}_P = 0$$

and

$$\begin{bmatrix} \mathbb{I}_1 \\ \mathbb{0} \end{bmatrix} \bar{\mathbb{x}}_B + \begin{bmatrix} \mathbb{P}_B \\ \mathbb{0} \end{bmatrix} \bar{\mathbb{x}}_P + \begin{bmatrix} \mathbb{0} \\ \mathbb{I}_s \end{bmatrix} \bar{\mathbb{s}}_B + \begin{bmatrix} \mathbb{R}_B \\ \mathbb{Q}_E \end{bmatrix} \bar{\mathbb{s}}_P = \begin{bmatrix} \mathbb{b}_B \\ \mathbb{0} \end{bmatrix}$$

$$\bar{\mathbb{x}}_B \geqslant \mathbb{0}, \quad \bar{\mathbb{s}}_B \geqslant \mathbb{0}, \quad \bar{\mathbb{x}}_P \ngeq \mathbb{0}, \quad \bar{\mathbb{s}}_P = 0.$$

In this notation, vectors $\bar{\mathbb{x}}_B$, $\bar{\mathbb{s}}_B$ from the base part of the phase I optimal solution and they are composed of decision and artificial variables. The resulting conclusion is that redundant equations have zero coefficients with all decision variables x_j and simultaneously zero RHS at the end of phase I computations.

Application of this reduction procedure requires to solve the problem with all constrains including potentially reduced. Sometimes it is worth doing, especially when the given set of constraints is used repeatedly.

<u>§4. Redundant and conditionally redundant variables[6/].</u>

A. It is obvious that variable x_j, having constant value $d_j \geqslant 0$ for all solutions from set $\mathfrak{X}$, is the redundant variable in

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6/ In this section we use problems discussed in chapter 5 of P.G. Leunberger work [5].

the problem /14/ - /16/. This is accomplished by formal condition $x_j \equiv d_j$, which stated in an initial formulation of the problem /14/ - /16/, automatically excludes variable x_j from further considerations. More general information on the constant sign of the variable x_j for all solutions also enables us to exclude this variable from the problem.

Definition 3

Variable x_p is the null variable of the system /15/ - /16/, if for every solution $x \in \mathcal{X}$, one has $x_p = 0$.

Theorem 7

If $\mathcal{X} \neq \phi$, then x_p is null variable iff there exists such $\mathbb{t} \in \mathbb{R}^m$ that

$$\begin{cases} \mathbb{t}^T \mathbb{A} \geqslant \mathbb{O}^T & /I/ \\ \mathbb{t}^T \mathbb{A}_p > 0 & /II/ \\ \mathbb{t}^T \mathbb{b} = 0 & /III/ \end{cases} \qquad /20/$$

where $\mathbb{A} = [\mathbb{A}_1 \, \mathbb{A}_2 \, \cdots \, \mathbb{A}_n]$ is [redacted] the system /15/ coefficient matrix with $\mathbb{A}_j$ denoting j-th column.

We shall prove only the sufficiency of conditions /I/ - /III/[7].

Let

$$d_j = \mathbb{t}^T \mathbb{A}_j \qquad /j = 1, \ldots, n/$$

/I/ and /II/ $\Rightarrow \forall_{1 \leq j \leq n} \; d_j \geqslant 0$ and $d_p > 0$ /what means that it has to be $\mathbb{t} \neq \mathbb{O}$ /.

7/ Complete proof can be found in [5].

$$\left.\begin{array}{r}/15/,\ \mathrm{II}\\ d_j \geq 0\end{array}\right\} \Rightarrow \left.\begin{array}{c}\sum_{j=1}^{n} d_j x_j = 0\\ x_j \geq 0,\ d_j \geq 0\end{array}\right) \Rightarrow \left.\begin{array}{c}\forall_{1 \leq j \leq n}\ d_j x_j = 0\\ x_p \geq 0,\ d_p > 0\end{array}\right) \Rightarrow x_p = 0$$

what completes the proof.

Corollary 2

If there is a homogeneous equation in system /15/ - /16/

$$\sum_{j \in J} a_{kj} x_j = 0 \text{ and } a_{kj} > 0 \text{ for } j \in J \qquad /21/$$

then variables x_j for $j \in J$ are null variables of system /15/ - /16/. It is possible to reduce this system by removal of equation "k" as unconditional and by removal of null variables x_j $(j \in J)$.

Suppose that system /15/ is presented in its basic form, assigning basic feasible solution $\mathbb{x}^B$. Then, after some regrouping of equations and variables, its coefficients matrix $\mathbb{A}_B$ and RHS have a property

$$\mathbb{A}_B = [\mathbb{I} \quad \mathbb{P}_B], \quad \mathbb{b}_B \geq \mathbb{0}$$

Satisfaction of conditions /I/ - /III/ leading to null variables, requires such vector $\mathbb{t} \neq 0$, that

$$\mathbb{t}^T \mathbb{A}_B \geq \mathbb{0}^T \Rightarrow [\mathbb{t}^T \quad \mathbb{t}^T \mathbb{P}_B] \geq [\mathbb{0}_1^T \ \mathbb{0}_2^T]$$

It means a replacement of the requirement $\mathbb{t} \neq \mathbb{0}$ by inequality $\mathbb{t} \geq \mathbb{0}$. But inequalities $\mathbb{t} \geq \mathbb{0}$, $\mathbb{b}_B \geq \mathbb{0}$ give rise to the equation $\mathbb{t}^T \mathbb{b}_B = 0$, which is condition /III/ iff

$$\forall_{1 \leq i \leq m} \quad b_i^B > 0 \qquad t_i = 0 .$$

Corollary 3

Lack of degeneracy in basic feasible solutions of system /15/ - /16/ implies nonexistence of null variables.

Note that significant practical implications has corollary 3. Seldom there are LP problems without at least one degenerated basic solution. According to corollary 3, null variables occur in LP problem rarely. Let's start to consider a group of positive variables now.

Definition 4

Variable x_j of the system /15/ - /16/ is positive variable iff in every solution $x \in \mathcal{X}$ one has $x_j > 0$.

Theorem 8 [5]

Suppose that the set $\mathcal{X} \neq \emptyset$. Variable x_s is a positive variable of the system /15/ - /16/ iff there exists such nonzero vector $t \in R^m$ that

1^o $\quad t^T A = d$ and $\begin{cases} d_j \geqslant 0, & \text{for } j \neq s \\ d_s = -1 \end{cases}$

2^o $\quad t^T b = -\beta \quad (\beta > 0)$.

Suppose that assumptions 1^o and 2^o are satisfied.

$$/15/ \Rightarrow \left.\begin{array}{c} \forall_{x \in \mathcal{X}} \; t^T A x = t^T b \\ 1^o \text{ and } 2^o \end{array}\right\} \Rightarrow x_s = \beta + \sum_{j \neq s} d_j x_j$$

Then

$$1^o \Rightarrow \forall_{x \in \mathcal{X}} \sum_{j \neq s} d_j x_j \geqslant 0 \Rightarrow \forall_{x \in \mathcal{X}} x_s > 0.$$

So, variable x_s is a positive variable of the system. A specific reduction possibility of system /15/ - /16/ results from a proof of theorem 8 sufficiency condition, presented above.

Theorem 9

If the following equation in system /15/ - /16/ exists:

$$\sum_{j=1}^{n} a_{kj} x_j = b_k$$

where $a_{kj} \geqslant 0$ $/j \neq s/$, $a_{ks} < 0$, $b_k < 0$, then after transformation to

$$x_s = \beta_s + \sum_{j \neq s} \alpha_j x_j , \qquad /22/$$

where $\beta_s = b_k : a_{ks}$, $\alpha_j = /-a_{kj}/ : a_{ks}$

the above relation is used as a substistution to problem /14/-/16/ in order to remove variable x_s from the objective function /14/ and constraints /15/, and to clear the system /15/ of the equation "k".

Note that nonzero lower bounds problem, well known in LP literature, is reduced to positive variables problem.

B. Now, we pay our attention to conditionally redundant variables and constraints, which are frequently used in different LP reduction procedures.

Definition 5

Variables taking zero values at every optimal solution of LP problem are called conditionally redundant.

The definition of the set $\mathcal{X}$, objective function and type of optimization play an important role in the conditionally redundant variables determination. Conditionally redundant variables can be removed from the problem before solving it, without effect on the choice of optimal solution. Unfortunately a simple method of the conditionally redundant variables determination does not exist. ~~Existance of simplex method by solution procedure for a system of m equations and n unknowns~~

Existence of such algorithm would imply replacement of a simplex method by solution procedure for a system of m equations and n unknowns in a case of one-point set of optimal solutions.

For any given solution we treat LP problem constraint as active, if it is satisfied as equality and we call it inactive, if it is satisfied as strong inequality.

Definition 6

LP problem constraint "k" is conditionally redundant, if it is inactive for every optimal solution.

From the above definition it follows that the constraint initially formulated as equation can not be conditionally redundant.

Duality of linear programming enables us to link a notion of conditionally redundant constraints with a notion of conditionally redundant variables.

To show this, we shall use a symetric dual problems. As a primal LP problem let us take the problem /1/ - /3/. Then, its dual problem is

$$y b \longrightarrow \min \qquad /23/$$

s.t.

$$\begin{cases} y A \geqslant c & /24/ \\ y \geqslant 0, & /25/ \end{cases}$$

where A is m x n matrix and y is "m" components row vector.

One of the fundamental theorems in duality theory is so called complementarity theorem.

Theorem 10

Satisfaction of the following equations is a necessary and sufficient condition for feasible solutions $\bar{x}$ and $\bar{y}$ of problems /1/ - /3/ and /23/ - /25/ respectively, to be its optimal solutions.

$$\bar{y}(b - A\bar{x}) = 0 \qquad /26/$$

$$(\bar{y}A - c)\bar{x} = 0. \qquad /27/$$

Frequently conditions /26/ - /27/ are replaced by the equivalent four implications:

$$\bar{y}_i > 0 \Rightarrow a_i\bar{x} = b_i, \qquad /28/$$

$$a_i\bar{x} < b_i \Rightarrow \bar{y}_i = 0, \qquad /29/$$

$$\bar{x}_j > 0 \Rightarrow \bar{y}A_j = c_j, \qquad /30/$$

$$\bar{y}A_j > c_j \Rightarrow \bar{x}_j = 0. \qquad /31/$$

Implications described by /29/ and /31/ can be understood in the following way. If in one of the dual problems /1/ - /3/, /23/- /25/ some constraint is conditionally redundant at a given optimal solution, then a dual variable conjugated with it, is also conditionally redundant. Implications /28/ and /30/ lead to conclusion that positive value of a decision variable at optimal solution implies activity of a dual constraint conjugated with it.

These four implications /28/ - /31/ we shall apply while analysing one of the reduction procedures in the next section.

§5. Two reduction procedures of LP problems

A. In this section we are going to present two reduction procedures of LP problems, developed by Polish authors. They are not so widely known as methods presented by A.L. Brearley, G. Mitra, H.P. Williams in [1] or by I.W.Joslowicz, J.M. Makarenko in [3].

The reduction procedure of the following problem is discussed in [1]:

$$cx \longrightarrow \max$$

s.t.

$$a_i x \leq b_i, \quad i \in J^+$$

$$a_i x \geq b_i \qquad i \in J^-$$
$$a_i x = b_i \qquad i \in J^=$$
$$d \leq x \leq g$$

The reduction procedure of the following problem is considered in [3]:

$$cx \longrightarrow \max$$

s.t.

$$Ax \leq b$$
$$0 \leq x \leq g$$

where c, b and g are positive vectors of proper dimensions, A is $m \times n$ nonnegative matrix.

As can be seen upper bounds play an important role in both cases. Procedures treated in this section have a lot in common with I.W.Joslowicz, J.M.Makarenko work [3]. In the first procedure one assumes that $A \geq 0$, $b \geq 0$. The second one is based on paper [5] and takes advantage of an auxilliary problem, formulated as LP problem with one constraint and lower and upper bounds for particular variables. It should be stressed that an assumption concerning positive values of all problem parameters is neglected in the second procedure, what is not the case with I.W.Joslowicz, J.M. Makarenko method.

Now we shall present an algorithm proposed by W.Radzikowski in [7]. In this procedure the notion of redundant constraints is utilized to some extend. It is not treated as a sufficient condition, what explains a necessity of trial and error approach. In practice Radzikowski's method is fairly effective, what is reflected by a small number of iterations needed to obtain optimal solution.

The problem /1/ - /3/ with all $b_i > 0$, $a_{ij} \geqslant 0$ and $\sum_i a_{ij} > 0$ is a starting point. The numerical process can be divided into the following stages:

Stage I : for all $1 \leqslant j \leqslant p$ we compute

$$d_{ij} = w \cdot \frac{a_{ij}}{b_i} \quad /i \in \mathfrak{J}/$$

where w is a small positive number, introduced in order to increase d_{ij} values.

Stage II: for all $1 \leqslant j \leqslant p$ we calculate

$$d_{r_{ij}} = \max \left\{ d_{ij} \mid i \in \mathfrak{J} \right\}$$

Stage III: we form the set $\mathfrak{R} = \{r_j\} \subseteq \mathfrak{J}$ and from the system /2/ we delete inequalities with numbers $i \notin R$.

Stage IV: we solve the problem:

$$f(x) = c\,x \longrightarrow \max \qquad /32/$$

s.t.

$$a_i x \leqslant b_i \quad i \in \mathfrak{R} \qquad /33/$$

$$x \geqslant 0 \qquad /34/$$

and denote its optimal solution[8/] by $\bar{x}$.

Stage V: on the ground of $\bar{x}$ we check if for all $i \in \mathfrak{J} - \mathfrak{R}$

$$a_i \bar{x} \leqslant b_i$$

YES - $\bar{x}$ is optimal solution of the set $\mathfrak{X}$

NO - go to stage VI.

Stage VI : we enlarge the problem /14/ - /16/ by these inequalities $a_i x \leqslant b_i$ with $i \in \mathfrak{J} - \mathfrak{R}$, where

$$a_i \bar{x} > b_i$$

8/ According to assumptions concerning b_i and a_{ij}, set $\mathfrak{X}$ is nonempty and bounded, hence optimal solution of /32/-/34/ does exist.

With so enlarged problem /32/ - /34/ we follow calculations of stage IV and later.

Remark 1

There is a limited number of returns to stage IV. In the worst case the last return would lead to enlarged problem equal to problem /1/ - /3/.

Remark 2

Just after first formulation of the reduced problem /32/ - /34/ it is possible on the grounds of the system /33/ to define redundant inequalities with indices belonging to $\mathfrak{J} - \mathfrak{R}$. To accomplish this one has to use formulas /11/, /11'/ or corollary 1 in more complicated cases.

C. Now some comments dealing with the presented above stages.

The quotient $b_i : a_{ij}$ defines a maximal value reached by variable x_j at the constraint "i". Because it is possible to have $a_{ij}=0$, one assumes /stage I/:

$$d_{ij} = w \cdot \bar{a}_{ij}$$

Therefore the quantity $d_{r_j j}$ /stage II/ can be defined as

$$d_{r_j j} = \min_{1 \leq i \leq m} \left\{ \frac{b_i}{a_{ij}} \;\middle|\; a_{ij} > 0 \right\}$$

what correspondes to the determination of the inequality, being the most active upper bound for x_j values. From this point of view, inequalities "i" $\notin r_j$ are redundant.

The first version of the reduced problem /stage III/ includes only inequalities being ~~the~~ most active upper bounds for some variable x_j.

The set $\mathfrak{X}$ is a subset $\bar{\mathfrak{X}}$ of the reduced problem solutions set. From the above it follows that if the optimal solution

$\bar{x} \in \bar{\mathcal{X}}$ /stage IV/ satisfies condition $\bar{x} \in \mathcal{X}$ /stage V/ it is also an optimal solution of the initial problem. On the other hand it is easy to show that initial problem inequalities, with the strongest influence over variables need not to define the optimal solution of the set $\mathcal{X}$.

D. At present 9/ we describe LP problems reduction procedure proposed by K. Zarychta [9].

We consider the problem /1/ - /3/, now called P, and its dual problem /23/ - /25/, now called D.

Apart from already defined in part A of section 2 notions of the sets $\mathcal{X}, \mathcal{X}_{/k}$, we shall introduce another one, while considering P problem:

$$z = \max_{x \in \mathcal{X}} cx \; ; \qquad \bar{\mathcal{X}} = \{ x \mid x \in \mathcal{X}, \; cx = z \}$$

$$z_{/k} = \max_{x \in \mathcal{X}_{/k}} cx \; ; \qquad \bar{\mathcal{X}}_{/k} = \{ x \mid x \in \mathcal{X}_{/k}, \; cx = z_{/k} \} .$$

So, $\mathcal{X}, z, \bar{\mathcal{X}}$ define respectively:

- feasible solutions set,
- maximal value of the objective function taken on the set $\mathcal{X}$,
- optimal solutions set of P problem.

Whereas $\mathcal{X}_{/k}, z_{/k}, \bar{\mathcal{X}}_{/k}$ denote respectively:

- feasible solutions set,
- maximal value of the objective function taken on the set $\mathcal{X}_{/k}$
- optimal solutions set of the problem created from P by deletion of the constraint:

$$a_k x \leq b_k$$

9/ This part of the section 5 is based on chapter 8 in [6]

Similarly, for the problem D we introduce:

$$\mathcal{Y} = \{ y \in R^m \mid y \geq 0,\ y\mathbb{A}_j \geq c_j \ (j=1,\dots,n) \}$$

$$w = \min_{y \in \mathcal{Y}} y b, \quad \bar{\mathcal{Y}} = \{ y \mid y \in \mathcal{Y},\ y b = w \}$$

$$\mathcal{Y}_{/r} = \{ y \in R^m \mid y \geq 0,\ y\mathbb{A}_j \geq c_j \ (j=1,\dots,n;\ j \neq s) \}$$

$$w_{/r} = \min_{y \in \mathcal{Y}_{/r}} y b$$

$$\bar{\mathcal{Y}}_{/r} = \{ y \mid y \in \mathcal{Y}_{/r},\ y b = w_{/r} \}$$

It is easy to prove correctness of the following lemmas.

Lemma 1

Three conditions

$$\bar{\mathcal{X}}_{/k} \subset \mathcal{X}_{/k}, \quad \bar{\mathcal{X}}_{/k} \subset \mathcal{X}, \quad \bar{\mathcal{X}}_{/k} \subset \bar{\mathcal{X}}$$

are mutually equivalent. Each of them implies that the constraint "k" $a_k x \leq b_k$ of the P problem is conditionally redundant.

Lemma 1'.

Three conditions

$$\bar{\mathcal{Y}}_{/k} \subset \mathcal{Y}_{/k}, \quad \bar{\mathcal{Y}}_{/k} \subset \mathcal{Y}, \quad \bar{\mathcal{Y}}_{/k} \subset \bar{\mathcal{Y}}$$

are mutually equivalent. Each of them implies that the constraint "r" $y\mathbb{A}_r \geq c_r$ of the D problem is conditionally redundant.

Lemma 2

If $\bar{\mathcal{X}} \subset \mathcal{H}$ and $x \in \mathcal{H} \Rightarrow a_k x < b_k$, then the constraint "k" of the P problem is conditionally redundant, and for every $\bar{y} \in \bar{\mathcal{Y}}$ it is $\bar{y}_s = 0$ /is a conditionally redundant variable/.

Lemma 2'

If $\bar{\mathcal{Y}} \subset \mathcal{N}$ and $y \in \mathcal{N} \Rightarrow y\mathbb{A}_r > c_r$, then the constraint "r" of the D problem is conditionally redundant, and for every $\bar{x} \in \bar{\mathcal{X}}$ it is $x_r = 0$.

These two last lemmas will be used in a reduction procedure. If it is easy to define $\mathcal{M} \supset \bar{\mathfrak{X}}$, such that

$$\sup_{x \in \mathcal{M}} a_k x < b_k$$

then the constraint $a_k x \leqslant b_k$ can be removed from the problem P, and one can assume $y_k = 0$ in the problem D.

Set $\mathcal{M}$ one proposes to define as

$$\mathcal{M}_t = \{ x \mid c x \geqslant t , d \leqslant x \leqslant g \} \qquad /35/$$

Where t is a lower bound on the maximal value of cx in P problem. One has to compute $\max_{x \in \mathcal{M}_t} a_k x$ and check if

$$\max_{x \in \mathcal{M}_t} a_k x < b_k$$

Determination of $\sup_{x \in \mathcal{M}_t} a_k x$ dependence on t parameter is confined to parametric LP problem solving

$$\max_{x \in \mathcal{M}_t} a_k x \equiv \begin{pmatrix} a_k x \to \max \\ c x \geqslant t \\ d \leqslant x \leqslant g \end{pmatrix} \qquad /36/$$

W. Grabowski [10/] showed, that parametric LP methods need not be used to solve /36/.

While defining the set $\mathcal{M}_t$ one does not require set $\{x \mid d \leqslant x \leqslant g\}$ be bounded, so it can be R^n. But quite often the constraints $d \leqslant x \leqslant g$ are already in P problem or can be derived from other constraints by, for example, stage II of W. Radzikowski procedure.

Anyway, smaller interval of variables changes, better results one can expect with just described method.

10/ See pp. 134 - 146 in [6].

Similarly, one defines set $\mathcal{N} \supset \bar{y}$ such that

$$\inf_{y \in \mathcal{N}} y\mathbb{A}_r > c_r$$

Then, the constraint "r" can be removed from D problem and variable x_r can be deleted from P problem, because $x_r = 0$

Here one proposes to define set $\mathcal{N}$ as:

$$\mathcal{N}_t = \{ y \mid y\mathbb{b} \geqslant t \, , \, \mathbb{f} \leq y \leqslant \mathbb{h} \} \qquad /37/$$

where t is a lower bound on maximal cx value, assumed in $\mathcal{M}_t$ set. To calculate $\inf_{y \in \mathcal{N}_t} y\mathbb{A}_r$ we solve

$$\min_{y \in \mathcal{N}_t} y\mathbb{A}_r \equiv \begin{pmatrix} y\mathbb{A}_r \rightarrow \min \\ y\mathbb{b} \geqslant t \\ \mathbb{f} \leq y \leq \mathbb{h} \end{pmatrix} \qquad /38/$$

E. We can conclude the part D considerations by the following procedure:

Preliminary stage:

for every k = 1, ..., m one calculates

$$p_k = \inf \{ t \mid \max_{x \in \mathcal{M}_t} a_k x < b_k \} \qquad /39/$$

From $\mathcal{M}_t$ definition, given by /35/ it follows, that for $t_1 \geqslant t_2$ it is

$$\max_{x \in \mathcal{M}_{t_1}} a_k x \leqslant \max_{x \in \mathcal{M}_{t_2}} a_k x$$

Hence p_k is lower boundary of t parameter with maximal value of $a_k x$ in /36/ smaller than b_k.

For every r = 1, ..., n we compute also

$$q_r = \inf \{ t \mid \min_{y \in \mathcal{N}_t} y\mathbb{A}_r > c_r \} \qquad /40/$$

q_r defines lower boundary of t parameter with minimal value $y\mathbb{A}_r$ in /38/, greater than c_r.

It is the final part of preliminary stage, providing p_k and q_r values necessary for the probable reductions performed in the next stage.

Reduction stage:

we built sequence $z_1 < z_2 < \dots < z_s$ of consecutive increasingly better lower eliminations of the objective function cx optimal value in P problem.

The reduction rule results immediately from the following theorem[11].

Theorem 11

Let z_s /$z_s \leqslant z$/ be an objective function cx value, obtained at "s" iteration of P problem solving, or stated in another way. It is a lower bound for an optimal value of the objective function. If $z_s > p_k$, then P problem constraint "k" is conditionally redundant and a conjugated variable y_k of D problem is conditionally redundant. If $z_k > q_r$, the conditionally redundant are: D problem constraint "r" and a conjugated variable x_r of P problem.

Finally, we would like to comment on some numerical aspects of the reduction procedure.

1°. It is not necessary to compute all p_k and q_r. It is enough to consider only some subsets of these values.

2°. It is possible to use other, lower bounds of t parameter, than calculated according to /39/, /40/ p_k, q_r values. On the other hand, application of infimum notion to p_k and q_r definition provides better reduction possibilities.

3°. Some new information can result in cubicoids $\{x \mid d \leqslant x \leqslant g\}$ and $\{y \mid f \leqslant y \leqslant h\}$ dimensions decrease, what provides an opportunity to compute p_k and q_r values from the beginning.

11/ Proof of this theorem can be found in [6], p. 127.

Appendix A

The problemm /18/-/19/ belongs to the class of LP problems with absolute - value functionals and unrestricted variables u_i. According to G.Hadley[1] suggestion, problem /18/-/19/ should be replaced by LP problem:

$$\sum_{i \in J} (u_i^+ - u_i^-) \longrightarrow \max$$

s.t.

$$(u^+ - u^-)^T A_j = 0 \qquad /j=1,\dots,n/$$
$$(u^+ - u^-)^T b = 0$$
$$(u^+)^T u^- = 0$$
$$u^+ \geq 0, \quad u^- \geq 0 .$$

To solve it simplex method with bounds on introducing to the basis of one of the variables u_i, if the other from the pair is present /u_i^+ or u_i^- / can be used.

In [8] it was shown that such approach is not correct, because it leads to local optimum instead of global one. It is obvious also, because maximum finding of concave objective function on a covex set needn't to give global optimum.

As far as problem /18/-/19/ is concerned, these disadvantages of Hadley's proposal are not important, because all we want to know is a consistency of the problem, not its optimal solution. And this is achieved by the presented proposal.

1/ G.Hadley "Linear programming", Addison-Wesley, 1963, p. 172, ex.5-12

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