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THE SIMPLEX ALGORITHM IN A COMPACT FORM

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1. Introduction

This paper deals with a variant of the simplex algorithm which is suitable for solving linear programs with inequality constraints. It has also the great advantage over the canonical form of the simplex method, that much less computer memory is needed to store the required numerical informations in the different steps of the algorithm. Also, considerable computation time is saved since smaller tables have to be calculated. In order to have a comparison between the two methods, consider the following linear programming problem;

$$-3\,x_8 - 4\,x_9 = x_1 = \text{minimum}$$

subject to the equations :

$$\left.\begin{array}{lllllllll}
x_2 & & & & & & -x_8 & +\,3\,x_9 & = 9 \\
 & x_3 & & & & & -x_8 & +\,2\,x_9 & = 4 \\
 & & x_4 & & & & +x_8 & +\,2\,x_9 & = 6 \\
 & & & x_5 & & & & +\,x_9 & = 3 \\
 & & & & x_6 & & +x_8 & +\,x_9 & = 4 \\
 & & & & & x_7 & -x_8 & +\,x_9 & = 1
\end{array}\right\} \quad (1)$$

Such that $x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 \geq 0$

The equations are assumed to be in canonical form and a feasible basic solution

$$x_2 = 9 \qquad x_3 = 4 \qquad x_4 = 6$$

$$x_5 = 3 \qquad x_6 = 4 \qquad x_7 = 1$$

$$\text{and} \quad x_8 = x_9 = 0$$

is assumed to be known at the beginning, in order to avoid complications (accompanied with the technique of the two phase method) at this stage where the main idea of the algorithm has to be developed at first.* The soltuion in the canonical form is given in the following tables.

	$(-x_1)$	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	1
	1	0	0	0	0	0	0	-3	-4	0
given		1						-1	3	9
Table			1					-1	2	4
<u>Step 0</u>				1				1	2	6
					1			0	1	3
(Table 1a)						1		1	1	4
							1	-1	-1	1
	(1)	(2)	(3)	(4)	(5)	(6)	(7)			

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* This assumption is satisfied in most practical cases, however, if it is not satisfied, then a new two phase method can be applied (see Memo. 485)

Step 1

(table 1b)

$(-x_1)$	x_2 ,	x_3 ,	x_4 ,	x_5 ,	x_6 ,	x_7 ,	x_8 ,	x_9 ,	1
1	0	2	0	0	0	0	-5	0	8
	1	-1.5					0.5	0	3
		0.5					-0.5	1	2
		-1	1				2	0	2
		-0.5		1			0.5	0	1
		-0.5			1		1.5	0	2
		0.5				1	-1.5	0	3
(1)	(2)		(4)	(5)	(6)	(7)		(3)	

Step 2

(Table 1c)

$(-x_1)$	x_2 ,	x_3 ,	x_4 ,	x_5 ,	x_6 ,	x_7 ,	x_8 ,	x_9 ,	1
1	0	-0.5	2.5	0	0	0	0	0	13
	1	-1.25	-0.25						2.5
		0.25	0.25					1	2.5
		-0.5	0.5				1		1
		-0.25	-0.25	1					0.5
		0.25	-0.75		1				0.5
		-0.25	0.75			1			4.5
(1)	(2)			(5)	(6)	(7)	(4)	(3)	

Step 3

(table 1d)

$(-x_1)$	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	1^*
1	0	0	1	0	2	0	0	0	14
	1		−4		5				5
			1		−1			1	2
			−1		2		1		2
			−1	1	1				1
		1	−3		4				2
			0		1	1			5
(1)	(2)	(6)		(5)		(7)	(4)	(3)	

which gives the optimal solution:

$$\left.\begin{aligned} &\text{objective function } x_1 = -14 \\ &\text{and} \quad x_2 = 5, \quad x_3 = 2, \\ &x_4 = 0, \quad x_5 = 1, \quad x_6 = 0, \\ &x_7 = 5, \quad x_8 = 2, \quad x_9 = 2. \end{aligned}\right\} \qquad (2)$$

A remarkable feature of the above tables is that a unity matrix is always present in each table, the columns of which are marked by their numbers in brackets written below the tables. However, a unity matrix does not contain numerical informations which need be stored in so much space; in the above example the unity matrix occupies 70% of the tables size.

In order to get rid of the unity matrix, we can rewrite the given problem (1) in the following form :

$$\left.\begin{aligned} &\text{Minimize} \quad x_1 = -3\,x_8 - 4\,x_9 \\ &\text{subject to:} \\ &x_2 = x_8 - 3\,x_9 + 9 \geq 0 \\ &x_3 = + x_8 - 2\,x_9 + 4 \geq 0 \\ &x_4 = - x_8 - 2\,x_9 + 6 \geq 0 \\ &x_5 = - x_9 + 3 \geq 0 \\ &x_6 = - x_8 - x_9 + 4 \geq 0 \\ &x_7 = x_8 + x_9 + 1 \geq 0 \\ &\text{and} \quad x_8 \geq 0, \; x_9 \geq 0 \end{aligned}\right\} \qquad (3)$$

We have thus expressed the linear programming problem in another form where the constraints are all inequalities in the two independent variables x_8 and x_9. In fact, the compact variant of the simplex method can be directly applied to linear programs with inequality constraints. At the present we shall assume that the solution obtained by giving the independent variables zero values satisfy all the given inequalities. If this is not the case, a new modified two phase method(*) is to be applied.

2. The compact table form

Consider the following linear program with inequality constraints:

Minimize the objective function

$$z = c_1x_1 + c_2x_2 + \cdots\cdots + c_jx_j + \cdots\cdots + c_nx_n + d$$

subject to the inequalities:

$$\left.\begin{array}{l}
a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1j}x_j + \cdots + a_{1n}x_n + b_1 \geqq 0 \\
a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2j}x_j + \cdots + a_{2n}x_n + b_2 \geqq 0 \\
\quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots\cdots + a_{ij}x_j + \cdots + a_{in}x_n + b_i \geqq 0 \\
\quad\vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mj}x_j + \cdots + a_{mn}x_n + b_m \geqq 0
\end{array}\right\} \quad (4)$$

and $\quad x_1, x_2, \cdots\cdots, x_j, \cdots\cdots, x_n \geqq 0$

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*) See Memo 485
"A new two phase method for linear programs with inequality constraints"

We introduce the variables $y_1, y_2, \ldots, y_i, \ldots, y_m$ and write the system in the form of the following table

	x_1	$x_2 \ldots$	$x_j \ldots$	x_n	1
$z =$	c_1	$c_2 \ldots$	$c_j \ldots$	c_n	d
$y_1 =$	a_{11}	$a_{12} \ldots$	$a_{ij} \ldots$	a_{1n}	b_1
$y_2 =$	a_{21}	$a_{22} \ldots$	$a_{ij} \ldots$	a_{2n}	b_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$y_i =$	a_{i1}	$a_{i2} \ldots$	$a_{ij} \ldots$	a_{in}	b_i
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$y_m =$	a_{m1}	$a_{m2} \ldots$	$a_{mj} \ldots$	a_{mn}	b_m

(Table 2)

The problem can be then interpreted in the following way:

> It is required to solve the system of table 2, where the dependent variables $y_1, y_2, \ldots, y_i, \ldots, y_m$ are expressed in terms of the independent variables $x_1, x_2, \ldots, x_j, \ldots, x_n$, such that all variables assume nonnegative values and the objective function z be minimum.

In order to obtain the optimal solution for this problem, the simplex algorithm will be applied in a number of steps (called "pivoting steps"). In each step one of the dependent variables is exchanged with one independent variable; the sets of dependent and independent variables will be thus always changing. After each step a so called "current solution" is read from the table. If this "current solution" is found to be optimal the algorithm is terminated; otherwise a new pivoting step is performed.

In the following, rules for reading the current solution from the table and conditions for feasibility and optimality of this solution will be explained. In section 3 and 4 the mechanism of the pivoting step is given in detail.

Reading the "current solution" from the table

A solution which satisfies the system of table 2 can be easily read from the table in the following way:

1. Each independent variable is given a zero value.
2. The values of objective function and the dependent variables are then read from the constants column of the table.

It is then obvious that the current solution will be feasible if the table satisfies the following feasibility condition.

Feasibility condition:

If the constants column of the table (2) consists of non-negative numbers (with the exception of the first term), then the current solution will be feasible; i.e. it satisfies all the given inequalities.

A *feasible* current solution is optimal if the following optimality condition is also satisfied.

Optimality condition:

If the coefficients of the independent variables in the objective function are non-negative (i.e. the numbers in the first row of the table with the exception of the constant term), then a feasible current solution will be optimal.*

Proof . Since the independent variables are to assume non-negative values, and these have positive coefficients in the objective function, it is obvious that the objective function will be minimum if these assume zero values. The current solution will be thus an optimal solution.*

3. The pivoting operation

The transformation of the table 2 such that the dependent variable $y_{\underline{i}}$ is exchanged with the independent variable $x_{\underline{j}}$ is known as a "pivoting operation". The row $\underline{i}$ of the variable $y_{\underline{i}}$ is called the "pivot row", while the column $\underline{j}$ of the variable $x_{\underline{j}}$ is called the "pivot column". The element $a_{\underline{ij}}$ will be referred to as the "pivot element" or simply the "pivot" of the pivoting operation; its choice defines completely what is required to be exchanged). The table will have the following form, before and after the pivoting operation with pivot $a_{\underline{ij}}$:

		$x_{\underline{j}}$	x_j	1	
z =	...	$c_{\underline{j}}$	.. c_j ..	d	
		. .	. .	. .	
$y_{\underline{i}}$ =	...	$a_{\underline{ij}}$	.. $a_{\underline{i}j}$ ..	$b_{\underline{i}}$	pivot row $\underline{i}$
y_i =		. . $a_{i\underline{j}}$ . .	. . a_{ij} . .	. . b_i . .	
		pivot column $\underline{j}$			

a

		$y_{\underline{i}}$	x_j	1	
z=	...	$c_{\underline{j}}$	.. c_j ..	d	
		. .	. .	. .	
$x_{\underline{j}}$ =	...	$a_{\underline{ij}}$	.. $a_{\underline{i}j}$ ..	$b_{\underline{i}}$	row $\underline{i}$
y_i =	. .	. . $a_{i\underline{j}}$ . .	. . a_{ij} . .	. . b_i . .	
		column $\underline{j}$			

b

Table (3) Pivoting with pivot $a_{\underline{ij}}$

* Nothing has yet been said about the uniqueness of this optimal solution.

It is now required to obtain rules for calculating the elements of the new table (3b) from the elements of the old one (3a). For this purpose solve the equation in the pivot row in (3a) for the independent variable $x_{\underline{j}}$. The equation is:

$$y_{\underline{i}} = a_{\underline{i}1}x_1 + a_{\underline{i}2}x_2 + \dots + \boxed{a_{\underline{i}\underline{j}}x_{\underline{j}}} + \dots + a_{\underline{i}j}x_j + \dots a_{\underline{i}n}x_n + b_{\underline{i}} \qquad (5\text{-a})$$

solving for $x_{\underline{j}}$ and rearranging the right hand side such that the variable $y_{\underline{i}}$ in the new equation be written in place of the variable $x_{\underline{j}}$ in the old one, we get

$$x_{\underline{j}} = \left(\frac{-a_{\underline{i}1}}{a_{\underline{i}\underline{j}}}\right)x_1 + \left(\frac{-a_{\underline{i}2}}{a_{\underline{i}\underline{j}}}\right)x_2 + \dots + \left(\frac{1}{a_{\underline{i}\underline{j}}}\right)y_{\underline{i}} + \dots + \left(\frac{-a_{\underline{i}j}}{a_{\underline{i}\underline{j}}}\right)x_j + \dots + \left(\frac{-a_{\underline{i}n}}{a_{\underline{i}\underline{j}}}\right)x_n + \left(\frac{-b_{\underline{i}}}{a_{\underline{i}\underline{j}}}\right) \qquad (5\text{-b})$$

comparing this with the equation $x_{\underline{j}}$ of the new table we get the following rules

$$a'_{\underline{i}\underline{j}} = 1/a_{\underline{i}\underline{j}} \qquad (6\text{-a})$$

$$a'_{\underline{i}j} = -a_{\underline{i}j}/a_{\underline{i}\underline{j}}, \quad b'_{\underline{i}} = -b_{\underline{i}}/a_{\underline{i}\underline{j}} \qquad (6\text{-b})$$

Substituting for $x_{\underline{j}}$ from equation (5-b) in the other equations of the table (including the z equation) we get, after arranging the variables as in equation (5-b)

$$z = (c_1 + c_{\underline{j}} \cdot a'_{\underline{i}1}) \cdot x_1 + \dots + \underline{(c_{\underline{j}}/a_{\underline{i}\underline{j}})y_{\underline{i}}} + \dots + (c_j + c_{\underline{j}} \cdot a'_{\underline{i}j})\, x_j + \dots$$
$$+ \cdot (c_n + c_{\underline{j}} \cdot a'_{\underline{i}n})x_n + (d + c_{\underline{j}} \cdot b'_{\underline{i}}) \qquad (5\text{-c})$$

and

$$y_i = (a_{i1} + a_{i\underline{j}} \cdot a'_{\underline{i}1})x_1 + \dots + \underline{(a_{i\underline{j}}/a_{\underline{i}\underline{j}})y_{\underline{i}}} + \dots + (a_{ij} + a_{i\underline{j}}a'_{\underline{i}j})x_j + \dots$$
$$\dots + (a_{in} + a_{i\underline{j}}a'_{\underline{i}n})x_n + (b_i + a_{i\underline{j}}b'_{\underline{i}}) \qquad (5\text{-d})$$

Comparing (5-c) and (5-d) with the equations z and y_i of the new table we get the following rules.

$$a'_{i\underline{j}} = a_{i\underline{j}}/a_{\underline{i}\underline{j}} \,,\quad c'_{\underline{j}} = c_{\underline{j}}\,/a_{\underline{i}\underline{j}} \qquad (6\text{-}c)$$

and

$$a'_{ij} = a_{ij}+a_{i\underline{j}}\cdot a'_{\underline{i}j},\quad b'_i = b_i + a_{i\underline{j}}\, b'_{\underline{i}}$$

$$c'_j = c_j + c_{\underline{j}}\cdot a'_{\underline{i}j} \,,\quad d' = d + c_{\underline{j}}\cdot\, b'_{\underline{i}} \qquad (6\text{-}d)$$

Rules (6-a), (6-b), (6-c), (6-d) can be summed up in the following way

__Table (4)__

the rules of the pivoting operation.

	pivot column		
rule d	rule c	rule d	
rule b	rule a	rule b	pivot row
rule d	rule c	rule d	

__rule a__ New pivot element = 1/old pivot element

__rule b__ New element in pivot row = - old element/old pivot element

__rule c__ New element in pivot column = old element/old " "

__rule d__ New element (neither in pivot row nor in pivot column)

= old element + (old element of pivot column in the same row) × (new element of pivot row in the same column).

For calculating the elements of the new table the following steps can be used; these are suitable for hand calculations, (for computer calculations a better arrangement is given in the flow chart and FORTRAN PROGRAM in the appendix)

1. Calculate new pivot row using rules a and b
2. Leaving the pivot column unchanged, calculate the rest of the table using rule d.
3. Calculate the pivot column using rule c.

4. The choice of the pivot element

Up till now we know how to read a current solution from the table and we know conditions for its feasibility, and conditions for its optimality. We know also how to transform the table by exchanging its variables in order to obtain new current solutions. It remains now to decide in which way are the pivoting operations to be performed (i.e. which variables are to be exchanged) such that we get a sequence of current solutions, terminating in an optimal solution.

The choice of the pivot element will be subject to the following two requirements :

1. Since our aim is to obtain a minimum value for the objective function, it will be reasonable to demand that after each pivoting operation the new value of the objective function (as read from the table 3-b, $z=d'$) be decreased; i.e.

$$\boxed{d' \leqq d} \qquad (7\text{-a})$$

2. Since the required optimal solution has to be also feasible, we shall require that the current solution obtained after each pivoting step be always feasible. At the start a feasible current solution is given; i.e. in table 3-a it is assumed that

$$b_1, b_2, \ldots, b_i \ldots, b_m \geqq 0$$

it is then required from table 3-b that

$$\boxed{b_1', b_2', \ldots, b_i', \ldots, b_m' \geqq 0} \qquad (7\text{-b})$$

From these requirements we get the following rules for choosing the pivot element; i.e. the choice of the pivot row and the pivot column.

Choice of pivot column :

If after a pivoting step it is found that all $c_j \geqq 0$, then the optimality condition is satisfied and the current solution read from the table is an optimal solution.*)

If one of the c_j is less than zero, then it is seen by examining the objective function, that giving the corresponding independent variable in the column j a value greater than zero will result in a better (i.e. less) value for the objective function. Thus , it must be taken away from the set of independent variables and must be made a dependent variable; which means simply, that the column j is to be taken as a pivot column.

If more than one c_j is less than zero, then we have to choose one of the corresponding columns as a pivot column. An ideal choice would be that which would result in a maximum decrease in the objective function after the pivoting step; but this would complicate the choice unnecessarily.

A convenient rule is to choose the column with the most negative c_j, this is justified by the following consideration: From equation (6-d) it is seen that the decrease in the objective function will be

$$d - d' = - c_j \cdot b_i' = |c_j| \cdot b_i'$$

So we can say that we can hope for a great improvement (i.e. decrease) in the objective function, if a c_j has a large absolute value

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* If all $c_j > 0$, then the current solution is a unique optimal solution; i.e. it is the optimal solution.

We have thus the following

<u>Rule for choice of pivot column $\underline{j}$:</u>

$$\boxed{\begin{array}{lll} \text{Choose } \underline{j} & \text{such that} & \\ & c_{\underline{j}} \leq c_j \text{ , for all } j & \\ \text{If } c_{\underline{j}} < 0 & \text{then choose pivot row} & \\ \text{If } c_{\underline{j}} \geq 0 & \text{then terminate the algorithm;} & \\ \multicolumn{3}{l}{\text{the current solution is then an optimal solution}} \end{array}} \qquad (8)$$

<u>Choice of pivot row</u>

We shall now use the second requirement (7-b) in order to obtain a rule for the choice of the pivot row $\underline{i}$ <u>after</u> we have chosen the pivot column $\underline{j}$.

For this purpose consider at first equation (6-b):

$$b'_{\underline{i}} = - b_{\underline{i}} / a_{\underline{i}\underline{j}}$$

It is seen that a negative pivot element has to be chosen if it is required to have $b'_{\underline{i}} \geq 0$, (since $b_{\underline{i}} \geq 0$). We have then to consider the other elements b'_i : from equation (6-d):

$$b'_i = b_i + a_{i\underline{j}} \cdot b'_{\underline{i}}$$

For the elements b'_i in the rows where $a_{i\underline{j}} \geq 0$, it is seen that $b'_i \geq 0$, since $b_i \geq 0$ and $b'_{\underline{i}} \geq 0$. For the elements b'_i in the rows where $a_{i\underline{j}} < 0$ we have the following condition :

$$\begin{aligned} b'_i &= b_i + a_{i\underline{j}} \cdot b'_{\underline{i}} \\ &= b_i + a_{i\underline{j}} \cdot \left(\frac{-b_{\underline{i}}}{a_{\underline{i}\underline{j}}} \right) \geq 0 \end{aligned}$$

$$\text{or} \qquad b_i \geq a_{i\underline{j}} \left(\frac{b_{\underline{i}}}{a_{\underline{i}\underline{j}}} \right) \qquad (9)$$

$$\text{or} \qquad \left(\frac{b_i}{a_{i\underline{j}}} \right) \leq \left(\frac{b_{\underline{i}}}{a_{\underline{i}\underline{j}}} \right)$$

(the sense of inequality is reversed because $a_{ij} < 0$)
We have thus the following
Rule for choice of pivot row i :

1. Determine the rows with $a_{ij} < 0$ 2. Among these row choose the row with the greatest value (or the least absolute value) of the characteristic quotients $Q_i = b_i / a_{ij}$

(10)

5. A solved example

To illustrate the compact simplex algorithm, the problem given in section 1 will be now solved in the compact form. The linear program as defined by equations (3) and its solution in the compact form is given by the following tables.

Step 0 (pivot column: x_9)

	x_8	x_9	1	Q
$x_1 =$	-3	-4	0	
$x_2 =$	1	-3	9	-3
$x_3 =$	1	-2	4	-2 pivot row
$x_4 =$	-1	-2	6	-3
$x_5 =$	0	-1	3	-3
$x_6 =$	-1	-1	4	-4
$x_7 =$	1	1	1	

Table 5 a

Step 1 (pivot column: x_8)

	x_8	x_3	1	Q
$x_1 =$	-5	2	-8	
$x_2 =$	-0.5	1.5	3	-6
$x_9 =$	0.5	-0.5	2	
$x_4 =$	-2	1	2	-1 pivot row
$x_5 =$	-0.5	0.5	1	-2
$x_6 =$	-1.5	0.5	2	-4/3
$x_7 =$	1.5	-0.5	3	

Table 5 b

Step 2

Table 5 c

	x_4	x_3 (Pivot column)	1	Q
$x_1 =$	2.5	-0.5	-13	
$x_2 =$	0.25	1.25	2.5	
$x_9 =$	-1.25	-0.25	2.5	-10
$x_8 =$	-0.5	0.5	1	
$x_5 =$	0.25	0.25	0.5	
$x_6 =$	0.75	-0.25	0.5	-2 (Pivot row)
$x_7 =$	-0.75	0.25	4.5	

Step 3

Table 5 d

	x_4	x_6	1
$x_1 =$	1	2	-14
$x_2 =$	4	-5	5
$x_9 =$	-1	1	2
$x_8 =$	1	-2	2
$x_5 =$	1	-1	1
$x_3 =$	3	-4	2
$x_7 =$	0	-1	5

Optimal Solution

objective function

$x_1 = -14$

and:

$x_2 = 5$

$x_3 = 2$

$x_4 = 0$

$x_5 = 1$

$x_6 = 0$

$x_7 = 5$

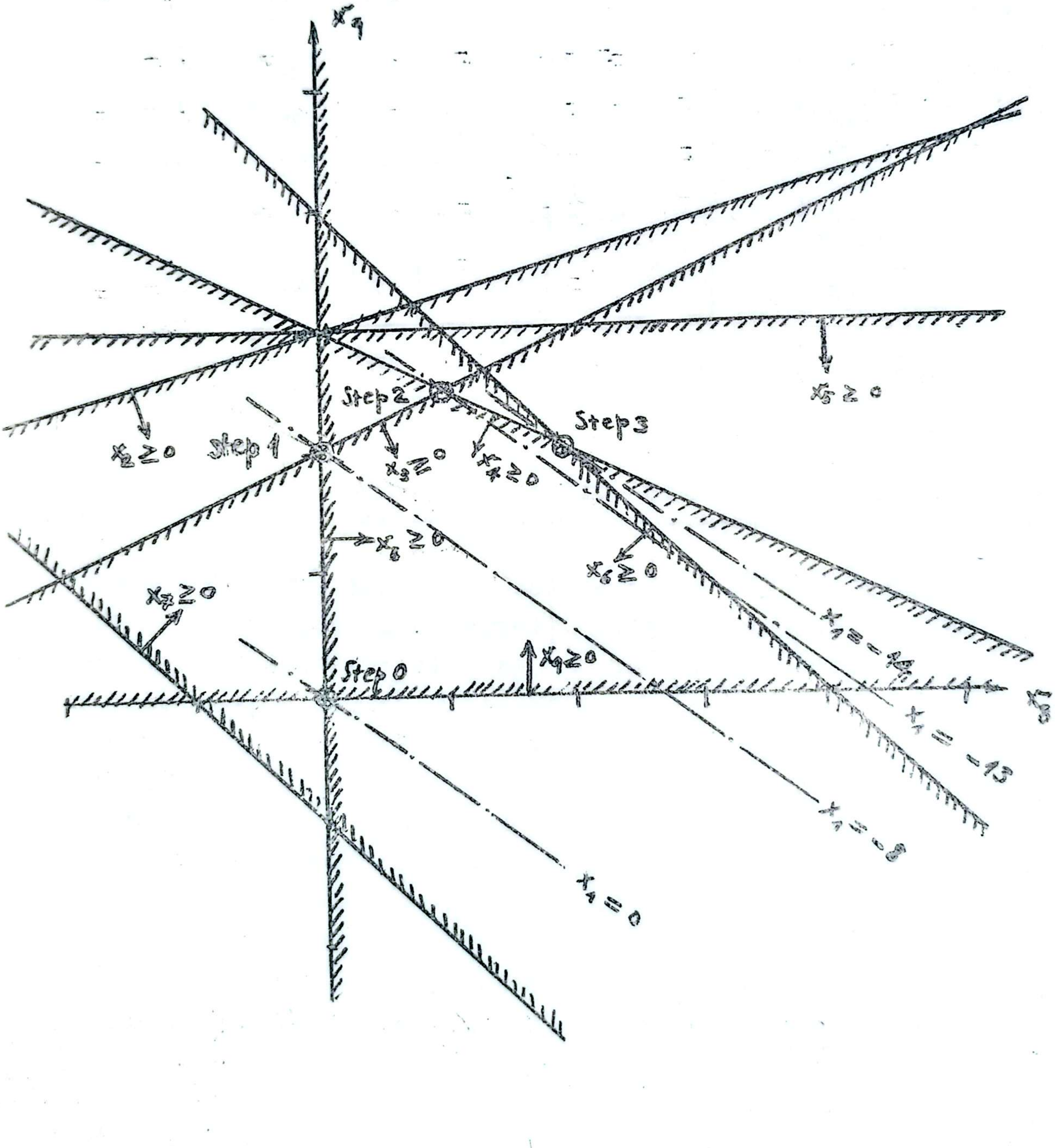
$x_8 = 2$

$x_9 = 2$

6. Geometrical interpretation of the algorithm

In order to have a geometrical representation of what happens to the Current Solution after each pivoting step during the algorithm, consider the example which we have solved in section 5. The system of inequalities can be represented by the following diagram; since each inequality is expressed in terms of the two independent variables x_8, x_9, these can be taken as our axes. The regions where each inequality is satisfied is indicated in the diagram, while the hatched area is the region where each inequality is satisfied; our aim is then to determine the "lowest" point in this polygon; where the terms "low" and "high" are defined by the objective function; the points on the line $x_1 = 0$ are thus considered to be on the same "horizontal" level.

The whole polygon lies (in this case) "below" this horizontal level.



The first Current Solution (step 0) is obtained by putting $x_8 = 0$, $x_9 = 0$; i.e. it can be represented by the origin. The assumption that the first Current Solution must be feasible means that the origin must belong to the region where the inequalities of the system are satisfied. Since the polygon lies "below" this point, it is clear that we must move to another "lower" point. In our case we have two possible ways to do this, namely to move either along the side $x_9 = 0$ or along the side $x_8=0$, the second side is chosen because it is "steeper". (This is the geometrical interpretation of the rule for choosing the pivot column). The second question is now, to which point are we to move ? Since the feasibility condition implies that the new current solutions is to be feasible it is clear that we have to move to a new corner of the polygon, or in other words we have to represent the system in two new axes meeting in that corner, namely the axes $x_8 = 0$, $x_3 = 0$.

If the objective function is also represented in these new axes, it will be seen that if both positive sides of the axes lie "above" the new "horizontal" level then the current solution is optimal; if not, a second step must be performed. The second step leads to the current solution given by the origin of the axes $x_4 = 0$, $x_3 = 0$. Examining the objective function it is seen that the positive side of the axes $x_4 = 0$ lies below the "horizontal" level, which means that we have to move along this axis to the point given by the origin of the new axes $x_4 = 0$, $x_6 = 0$. The current solution given by these points is seen to be optimal, since both positive sides of the axes $x_4 = 0$, $x_6 = 0$ lie above the new "horizontal" level defined by $x_1 = -14$.

7. Special situations in the Algorithm

In the above description of the algorithm we have always assumed the general case. However, in some problems one of the following situation might be encountered.

a. A non-unique optimal solution

Consider the same example solved in section 5, but replace the objective function by the following one:

$$x_1 = - x_8 - 2 \ x_9$$

After 2 steps the algorithm leads to the following table :

	x_4	x_3	1
$x_1=$	1	0	-6
$x_2=$	0.25	1.25	2.5
$x_9=$	-0.25	-0.25	2.5
$x_8=$	-0.5	0.5	1
$x_5=$	0.25	0.25	0.5
$x_6=$	0.75	-0.25	0.5
$x_7=$	-0.75	0.25	4.5

Table 6 a

non-unique optimal solution:

objective function $x_1 = -6$, and :

	$x_2 = 2.5$	$x_3 = 0$
$x_4 = 0$	$x_5 = 0.5$	$x_6 = 0.5$
$x_7 = 4.5$	$x_8 = 1$	$x_9 = 2.5$

It is seen that the coefficient of x_3 in the objective function is zero; this means that x_3 can be given a value other than zero, and the objective function remains unchanged. We have thus more than one optimal solution other than that given in table 6, let us try to exchange x_3 with one of the dependent variables, thus making it a new dependent variable, which gives it a value greater than zero. For this purpose the column of x_3 as a pivot column, then choose the pivot row according to rule (10) of section 4, then perform the pivoting step; this gives the following table

	x_4	x_6	1
x_1 =	1	0	-6
x_2 =	4	-5	5
x_9 =	-1	1	2
x_8 =	1	-2	2
x_5 =	1	-1	1
x_3 =	3	-4	2
x_7 =	0	-1	5

Table 6 - b

Another optimal solution

objective function $x_1 = -6$

and $x_2 = 5$ $x_3 = 2$

$x_4 = 0$ $x_5 = 1$ $x_6 = 0$

$x_7 = 5$ $x_8 = 2$ $x_9 = 2$

b. Unbound solution

Consider the following example

Step 0

	x_5	x_6	1
x_1 =	-3	-4	0
x_2 =	1	-3	9
x_3 =	1	-2	4
x_4 =	1	1	1

Step 1

	x_5	x_3	1
x_1 =	-5	2	-8
x_2 =	-0.5	1.5	3
x_6 =	0.5	-0.5	2
x_4 =	1.5	-0.5	3

Step 2

	x_2	x_3	1
x_1 =	10	-13	-38
x_5 =	-2	3	6
x_6 =	-1	1	5
x_4 =	-3	4	12

Tables 7 An example for an unbound solution

Applying the rules for pivot choice to the table after step 2, we find that according to rule (8) the column of x_3 has to be taken as a pivot column. But when we try to apply rule (10), we find that we are faced with the situation that in the pivot column all other elements are positive; which make it impossible to choose any pivot row.

This situation has the following meaning :

$$
\begin{aligned}
x_1 &= -15t - 38 \qquad (\text{putting } x_2 = 0,\ x_3 = t) \\
x_5 &= 3t + 6 \qquad\qquad (11) \\
x_6 &= t + 5 \\
x_4 &= 4t + 12
\end{aligned}
$$

This means that any positive value t of x_3 defines a feasible solution and for a larger value of t we have a better (less) value for the objective function x_1; i.e. we have an unbound solution.

We have thus the following

<u>rule for unbound solution</u>:

If a column j is found with $c_j < 0$ and $a_{ij} \geq 0$ (for all i), then the linear program has unbound solutions. To construct a ray of unbound solutions give the variable in the column j a value $t \rightarrow \infty$, and put the variables in the other columns equal to zero, then read the values of the other variables (including the objective function) from the table as function of the parameter t, where t can be arbitrarly large.

./.

e. A degenerated current solution

$1 + (-2 \times 1)$

Consider the following example :

	x_7	x_8	1	Q
$x_1 =$	-1	-1	0	
$x_2 =$	1	-1	2	-2
$x_3 =$	-1	-2	4	-2
$x_4 =$	1	-2	4	-2
$x_5 =$	0	-1	2	-2
$x_6 =$	-1	1	1	

	x_7	x_2	1	Q
$x_1 =$	-2	1	-2	
$x_8 =$	1	-1	2	
$x_3 =$	-3	2	0	0
$x_4 =$	-1	2	0	0
$x_5 =$	-1	1	0	0
$x_6 =$	0	-1	3	

Table 8 a An example for a degenerated solution

After the first pivoting step, we have to choose the column of x_7 as a pivot column, then trying to choose a pivot row we find that three characteristic quotients are equal to zero. We have thus to choose our pivot row among these rows; for example choose the row of x_4, the pivoting step yields the following table :

	x_4	x_2	1
$x_1 =$	2	-3	-2
$x_8 =$	-1	1	2
$x_3 =$	3	-4	0
$x_7 =$	-1	2	0
$x_5 =$	1	-1	0
$x_6 =$	0	-1	3

Table 8 b - Degeneracy

We notice that :

1. The objective function has not increased.
2. The table gives the same current solution.
3. If we choose a new pivot, we find that there are still zero characteristic quotients.

This has the following consequences :

1. The aim of the pivoting step (to obtain a better current solution) is not realized.
2. We are in danger of remaining in the same current solution for several pivoting steps, so that it is possible that after some steps an old table appears again; the result is that the algorithm will cycle indefinitely through the same set of tables without ever reaching an optimal solution.

This situation, which is called "degeneracy " has to be treated by special means (*), in order to avoid cycling of the algorithm.

8. Proof of finiteness of the algorithm.

We have now to prove that the algorithm actually leads to an optimal solution (or an unbound solution) after a finite number of pivoting steps.

Under the assumption that during the algorithm no degeneracy is encountered (**), the finiteness of the algorithm can be proved by the following argument:

Since no degeneracy is to be encountered, all calculated characteristic quotients b_i/a_{ij} are different from zero, i.e. $b_i \neq 0$. Examining the last equation(6-d), it is then seen that

(*) See Memo. No.

(**) The Proof fails if a degenerated current solution is encountered.

the objective function must strictly decrease after each pivoting step. Since the situation of the table is totally determined by the particular distribution of the variables in the 2 groups of dependent and independent variables, and there is a finite number of such distributions, therefore there is only a finite number of possible tables. And since we never return to the same table (because the objective function is always decreasing), therefore we must arrive to an optimal table (or to a table which defines an unbound solution) after a finite number of steps.

9. The dual linear program and a constructive proof of the duality theorem

Consider the linear program (4), which will be called the primal problem; this is represented by the table (9-a).

In order to construct the dual problem to the above primal, we have at first to rewrite the program (4) in the standard form :

primal problem

Minimize :

$$z = c_1x_1+c_2x_2 + \dots + c_jx_j + \dots + c_Nx_N$$

subject to :

$$\left.\begin{array}{l}
a_{11}x_1+a_{12}x_2+ \dots + a_{1j}x_j+ \dots + a_{1N}x_N \geqq -b_1 \\
a_{21}x_1+a_{22}x_2+ \dots + a_{2j}x_j+ \dots + a_{2N}x_N \geqq -b_2 \\
\vdots \\
a_{i1}x_1+a_{i2}x_2+ \dots + a_{ij}x_j+ \dots + a_{iN}x_N \geqq -b_i \\
\vdots \\
a_{M1}x_1+a_{M2}x_2+ \dots + a_{Mj}x_j+ \dots + a_{MN}x_N \geqq -b_M
\end{array}\right\} \quad (12)$$

such that

$$x_1 , x_2 , \dots , x_j, \dots , x_N \geqq 0$$

According to the definition of duality(*), the dual problem to (12) is the following.

(*) See Memo. No. 483

"Basic principles in linear programming" dual problem

Maximize :

$$V = -b_1u_1 - b_2u_2 - \dots - b_iu_i - \dots - b_Mu_M$$

subject to

$$\left.\begin{array}{l} a_{11}u_1 + a_{21}u_2 + \dots + a_{i1}u_i + \dots + a_{M1}u_M \leqq c_1 \\ a_{12}u_1 + a_{22}u_2 + \dots + a_{i2}u_i + \dots + a_{M2}u_M \leqq c_2 \\ \vdots \\ a_{1j}u_1 + a_{2j}u_2 + \dots + a_{ij}u_i + \dots + a_{Mj}u_M \leqq c_j \\ \vdots \\ a_{1N}u_1 + a_{2N}u_2 + \dots + a_{iN}u_i + \dots + a_{MN}u_M \leqq c_N \end{array}\right\} \quad (13)$$

such that u_1 , u_2, ... , u_i, ... , $u_M \geqq 0$

which can be written in the following form :

Maximise

$$V = b_1(-u_1) + b_2(-u_2) + \dots + b_i(-u_i) + \dots + b_M(-u_M) \qquad (14)$$

subject to

$$\begin{array}{l} v_1 = a_{11}(-u_1) + a_{21}(-u_2) + \dots + a_{i1}(-u_i) + \dots + a_{M1}(-u_M) + c_1 \geqq 0 \\ v_2 = a_{12}(-u_1) + a_{22}(-u_2) + \dots + a_{i2}(-u_i) + \dots + a_{M2}(-u_M) + c_2 \geqq 0 \\ \vdots \\ v_j = a_{1j}(-u_1) + a_{2j}(-u_2) + \dots + a_{ij}(-u_i) + \dots + a_{Mj}(-u_M) + c_j \geqq 0 \\ \vdots \\ v_N = a_{1N}(-u_1) + a_{2N}(-u_2) + \dots + a_{iN}(-u_i) + \dots + a_{MN}(-u_M) + c_N \geqq 0 \end{array}$$

such that u_1, u_2, ... u_i ... , $u_M \geqq 0$

This is represented in the table (9 -b)

	x_1	x_2	....	x_j	...	x_N	1
$z =$	c_1	c_2	....	c_j	...	c_N	d
$y_1 =$	a_{11}	a_{12}	....	a_{1j}	...	a_{1N}	b_1
$y_2 =$	a_{21}	a_{22}	....	a_{2j}	...	a_{2N}	b_2
.	.	.		.		.	.
.	.	.		.		.	.
$y_i =$	a_{i1}	a_{i2}	....	a_{ij}	...	a_{iN}	b_i
.	.	.		.		.	.
.	.	.		.		.	.
$y_M =$	a_{M1}	a_{M2}	....	a_{Mj}	...	a_{MN}	b_M

Table (9-a)
The primal problem

	$v_1 =$,	$v_2 =$,	....	$v_j =$,	...	$v_N =$,	$V =$
1	c_1	c_2	....	c_j	...	c_N	d
$(-u_1)$	a_{11}	a_{12}	....	a_{1j}	...	a_{1N}	b_1
$(-u_2)$	a_{21}	a_{22}	....	a_{2j}	...	a_{2N}	b_2
.	.	.		.		.	.
.	.	.		.		.	.
$(-u_i)$	a_{i1}	a_{i2}	....	a_{ij}	...	a_{iN}	b_i
.	.	.		.		.	.
.	.	.		.		.	.
$(-u_M)$	a_{M1}	a_{M2}	....	a_{Mj}	...	a_{MN}	b_M

Table (9-b)
the dual problem

The elements of the tables of the primal and dual problems are the same. Only the method of reading the tables is different; in the primal table (9-a) the dependent variables

$z, y_1, y_2, \ldots, y_i, \ldots, y_M$ are expressed in terms of the independent variables $x_1, x_2, \ldots, x_j \ldots, x_N$, while in the dual table (9-b) the dependent variables $v_1\ v_2, \ldots, v_j, \ldots, v_N$; V are expressed in terms $u_1, u_2, \ldots, u_i, \ldots, u_M$.

A current solution of the dual problem can be read from the table (9-b) in the following way :

Put the independent variables

$$u_1 = u_2 = \ldots, = u_M = 0 \tag{13-a}$$

then the values of the dependent variables will be

$$v_1 = c_1,\ v_2 = c_2,\ \ldots,\ v_j = c_j, \ldots, v_N = c_N \tag{13-b}$$

and the value of the objective function

$$V = d \tag{13-c}$$

From which we deduce the following :

Feasibility condition for the dual problem

A current solution of the dual problem is feasible if in the corresponding table

$$c_j \geqq 0 \text{ for } j = 1, 2, \ldots, N \tag{14}$$

The following optimality condition can be proved in a similar way as in section 2.

Optimality condition for the dual problem

A feasible current solution is optimal if in the corresponding table

$$b_i \geqq 0 \text{ for } i = 1, 2, \ldots, M \tag{15}$$

We have thus the following relation between the primal and the dual problems:

Primal problem	dual problem
Current solution is <u>feasible</u> if $b_i \geqq 0 \quad i=1,2,\ldots,M$	Current solution is <u>feasible</u> if $c_j \geqq 0 \quad j=1,2,\ldots,N$
Feasible current solution is <u>optimal</u> if $c_j \geqq 0 \quad j=1,2,\ldots,N$	Feasible current solution is <u>optimal</u> if $b_i \geqq 0 \quad i=1,2,\ldots,M$

<u>Table 10 feasibility and optimality conditions in the primal and dual problems</u>

Consider now the case of an <u>unbound solution for the primal problem</u>; what does this mean for the dual problem ?

From section 7 we know that the primal problem has an unbound solution if in a column $\underline{j}$ it is found that

$$c_{\underline{j}} \leqslant 0 \qquad (16)$$

and $a_{i\underline{j}} \geqq 0 \quad$ (for all i)

The corresponding equation in the column $\underline{j}$ of the dual table will be

$$v_{\underline{j}} = c_{\underline{j}} + a_{1\underline{j}}(-u_1) + a_{2\underline{j}}(-u_2) + \ldots + a_{i\underline{j}}(-u_i) + \ldots + a_{M\underline{j}}(-u_M)$$

$$= c_{\underline{j}} - (a_{1\underline{j}}u_1 + a_{2\underline{j}}u_2 + \ldots + a_{i\underline{j}}u_i + \ldots + a_{M\underline{j}}u_M) \qquad (17)$$

Since $a_{i\underline{j}} \geqq 0$ it is clear that for any non-negative values of the variables $u_1, u_2, .., u_i, ..., u_M$

$$v_j \leq c_j < 0 \tag{18}$$

i.e. the system of inequalities of the dual problem can never be satisfied by any non-negative values of the variables. Which means that the dual problem has no feasible solution.

We have thus proved the following theorems

Theorem 1 a:

If the primal problem has an unbound solution then the dual problem has no feasible solution.

Since the primal problem is the dual of the dual problem, the following theorem 1 b: is also true.

Theorem 1 b:

If the dual problem has an unbound solution then the primal problem has no feasible solution.

If the primal problem has a feasible solution with a value = d for the objective function z which may be optimal or not optimal, then remove the inequalities of the dual problem which are read from the columns with $c_j < 0$, the result is a reduced dual problem. The feasible solution of the primal is then seen to be an optimal solution for the reduced dual problem with a value = d for the objective function V. It is clear that by introducing the inequalities which we have removed from the dual problem, we must get a lower (i.e. less better) value for the objective function V of the dual, which means that ???

$$d = Z_{\text{feasible}} \geqq V_{\text{optimal}}$$

This can be completed to the following relation

Theorem 2

$$\underbrace{Z_{\text{feasible}} \geqq Z_{\text{optimal}}}_{\text{solution of primal problem}} \geqq \underbrace{V_{\text{optimal}} \geqq V_{\text{feasible}}}_{\text{solution of dual problem}}$$

Pivoting in the dual table

It can be shown that in order that the dependent variable v_j may be exchanged with the independent variable u_i a pivoting operation according to the same rules (6-a, b, c, d) can be used. The result will be the table (9-c).

	$v_1=$,	$v_2=$,		$u_i=$,		$v_N=$	1
	c'_1	c'_2	...	c'_j	...	c'_N	d'
$(-u_1)$	a'_{11}	a'_{12}	...	a'_{1j}	...	a'_{1N}	b'_1
$(-u_2)$	a'_{21}	a'_{22}	...	a'_{2j}	...	a'_{2N}	b'_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$(-v_j)$	a'_{i1}	a'_{i2}	...	a'_{ij}	...	a'_{iN}	b'_i
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$
$(-u_M)$	a'_{M1}	a'_{M2}	...	a'_{Mj}	...	a'_{MN}	b'_M

Table 9-c pivoting in the dual table

This fact will be used for the following

1) To establish a constructive proof for the duality theorem
2) To obtain another method of solution of the linear program, which we will call " The dual method of solution ".

Constructive proof of the duality theorem :

We want now to prove the following important theorem for linear programming :

Duality theorem : (Theorem 3)

If solutions to the primal and dual problems exist, then both problems have optimal solutions such that the optimal values of the objective functions are the same.

Proof:

Since the primal problem has a feasible solution, then we can construct a table with a feasible current solution, for example by solving a phase 1 problem.(*) Starting from this table we can use the method of this paper to obtain an optimal solution for the primal problem.

This will be obtained after a number of pivoting operations resulting in a sequence of tables. If we read these tables in the dual way we find that in fact we have at the same time performed a number of pivoting operations for the dual problem; the final table of both problems being the same. Since the final table of the primal problem satisfies both feasibility and optimality conditions(**), but these are the same conditions for the dual problem. Which means that the final table represents an optimal solution also for the dual problem. Since the values of the objective functions in both optimal solutions are given by the same element d of the final table, it is clear

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(*) See Memo. No.483

(**) The primal problem is assumed to have no unbound solutions. Since an unbound solution for the primal means that the dual problem has no feasible solution.

that both optimal solutions have the same optimal value for their objective functions.

The relation between the solutions of the primal and dual problems is given in the following table :

primal problem		dual problem
unbound solution	$\longrightarrow$	no feasible solution
no feasible solution	$\longleftarrow$	unbound solution
a finite optimal solution	$\longleftrightarrow$	a finite optimal solution
(same value for the objective functions)		

Table (11) The relation between the solutions of the primal and dual problems.

The relation between the values of the variables in the optimal solutions of the primal and the dual problems can be seen through the following considerations:

1) Comparing tables (9-a) and (9-b) of the primal and dual problems we see that :

   a) Corresponding to the independent variables

$$X_1 , X_2 , \ldots , X_j , \ldots\ldots , X_N$$

   of the primal problem, we have the dependent variables

$$V_1 , V_2 , \ldots , V_j , \ldots , V_N$$

of the dual problem such that

$$\underline{X_j \text{ corresponds to (or is dual to) } V_j} \qquad (19\text{-a})$$

i.e. they are written in the same place in both table

b) Simelarly, the dependent variables of the primal problem

$$y_1 , y_2 , \ldots , y_i , \ldots , y_M$$

correspond to the independent variables of

$$u_1 , u_2 , \ldots , u_i , \ldots , u_M$$

of the dual problem such that

$$\underline{y_i \text{ corresponds to (or is dual to) } u_i} \qquad (19\text{-b})$$

i.e. they are written in the same place in both tables

2) The effect of a pivoting operation with a pivot element a_{ij} in both tables is to exchange the variable X_j with y_i in the primal table, and at the same time the variables V_j and u_i in the dual table are exchanged. The correspondence relations (19-a,b) are thus conserved after any pivoting operation.

3) Consider now the table of the optimal solution of the primal problem, which is an optimal table of the dual problem. Since this table is obtained by performing a finite number of pivoting operations on the table (9-a) or (9-b), therefore the correspondence relations (19-a,b) between the variables of the primal and dual problems are still satisfied.

According to the rules of reading the optimal solutions of both problems from this optimal table, we get the following relation between the values of the variables in the optimal solutions

$$\begin{aligned}
X_j &= 0 \longrightarrow V_j (\geqq) 0 \\
V_j &= 0 \longrightarrow X_j (\geqq) 0 \qquad (20) \\
y_i &= 0 \longrightarrow u_i (\geqq) 0 \\
u_i &= 0 \longrightarrow y_i (\geqq) 0
\end{aligned}$$

??

These relations(*) can be stated in the form of the following important theorem, which is the connecting link between the primal and dual problems.

Theorem 4 :

The variables which assume values greater than zero in the optimal solution of the primal problem are the dual variables to the variables with zero values in the optimal solution of the dual problem.

10. The compact dual method of solution

We have seen that the method discussed in section 2,3,4 can be applied to any linear program with inequality constraints if the initial current solution is feasible. In section 9 we have seen that according to the duality theorem the optimal solution of a given problem can be obtained (using theorem 4) if the optimal solution of the dual problem is known. Using these facts we are able to solve any linear program if it happens that the initial solution of the dual

*) The equality sings in equations (20) are valid only in the case of degenerated solution.

problem (corresponding to the initial currents solution of the primal) is feasible; i.e. if the initial solution of the primal problem satisfies the optimality condition (but not the feasibility condition).

We can thus start solving the dual problem untill one of the following situations is ecountered

1) The dual problem has an unbound solution; then we know according to theorem 1b that the primal problem has no feasible solution.

2) The dual problem has an optimal solution; then we know according to the duality theorem that the primal problem has an optimal solution which can be obtained by using the rules (20) according to theorem 4.

However, since we are not interested in the current solutions and the optimal solution of the dual problem, we shall always read the table in the primal way (9-a), keeping in mind while we choose the pivot element that we are in fact solving the dual problem (9-b). If we are to do this, then the following rules for choosing the pivot element must be used

1) <u>Choice of pivot row</u>

Choose the row with the most negative constant term

$$b_{\underline{i}} \leq b_i \quad \text{for all } i \tag{21}$$

If all $b_i \geq 0$ then the current solution is optimal.

2) <u>Choice of pivot column</u>

In the pivot row determine the elements

$$a_{\underline{i}j} > 0 \tag{22-a}$$

For these elements calculate the characteristic quotients

$$Q_j = C_j \;/\; a_{\underline{i}j} \qquad (22\text{-b})$$

The column $\underline{j}$ corresponding to the smallest characteristic quotient is then chosen as pivot column

$$\left.\begin{array}{l} Q_{\underline{j}} \leqq Q_j \\ \text{for all } j \text{ according to (22-b)} \end{array}\right\} \qquad (22\text{-c})$$

The pivoting operation is then performed according to the rules (6-a,b,c,d).

This method will be called the <u>compact dual method of solution</u>. Problems of the type of the nutrition problem can always be solved by using this method.

<u>An example</u> :

Using the compact dual method of solution for solving the nutrition problem which we have already solved by using the dual method of solution for the standard simplex method (see section 11 of Memo. No.43 on "The basic principles of linear programming").

The problem has been expressen in the standard form with inequality constraints:

Minimize

$$Z = 5X_1 + 2X_2 \qquad (23\text{-a})$$

Subject to the inequalities

$$\begin{aligned} X_3 &= 3X_1 + X_2 - 8 \geqq 0 \\ X_4 &= X_1 + 2X_2 - 12 \geqq 0 \\ X_5 &= 3X_1 + 2X_2 - 6 \geqq 0 \end{aligned} \qquad (23\text{-b})$$

which can be written in the compact table (12-a). After two pivoting operations (12-b),(12-c) the following optimal solution is obtained :

$$Z = 15.2$$

$$\text{and} \quad X_1 = 0.8 \qquad X_2 = 5.6 \qquad (24)$$

Step 0

Table (12-a)

		X_1	X_2	1	
		5	2	0	
X_3	=	3	1	-8	
X_4	=	1	2	-12	← pivot row
X_5	=	3	2	-6	
Q_i	=	5	1		

Step 1

Table (12-b)

		X_1	X_4	1	
		4	1	12	
X_3	=	2.5	0.5	-2	← pivot row
X_2	=	-0.5	0.5	6	
X_5	=	2	1	6	
Q_i	=	1.6	2		

Step 2

Table (12-c) **The optimal solution**

		X_3	X_4	1
		1.6	0.2	15.2
X_1	=	0.4	-0.2	0.8
X_2	=	-0.2	0.6	5.6
X_5	=	0.8	0.6	7.6

11. Excersises

1) Prove the following " uniqueness condition " for an optimal solution :

   An optimal solution is unique if the coefficients of the objective function are all greater than zero

   i.e. $c_j > 0$ for all j.

2) Consider the following rule for choosing the pivot column:

   For each column j with $c_j < 0$ choose a prospective pivot element a_{ij} (according to the rule (10) of section 4) then calculate the improvement (decrease) in the value of the objective function if the pivoting operation is to be performed with this pivot element; as a final pivot element choose the one which would give the maximum improvement in the objective function.

   a) How many more operations are required by this rule in comparison with the rule (8) of section 4.

   b) Does this rule guarantee a minimum number of pivoting steps to be performed untill the optimal solution is reached?

   c) Does the rule (8) guarantee a minimum number of pivoting operations?

   (Construct graphical examples to support your arguments)

3)

   a) Prove the following method for reducing the size of the table during the algorithm :

      If a row in the table has no negative elements then this row can be cancelled.

b) What is the geometrical interpretation of this rule?

c) Solve the example of section 5, using this rule; how much computation time has been saved?

d) Why is it not practical to use this rule for computer programs?

Is the rule usefull for problem solved by hand calculation?

4) What is the geometrical interpretation of the special situations of the algorithm discussed in section 7.

5) For the example of a non-unique optimal solution in section 7 (table 6 a) write down a representation for all optimal solutions.

6) Determine all optimal solutions of the following program and represent them graphically.

	X_5	X_6	1
$X_1=$	1	-3	2
$X_2=$	1	-2	4
$X_3=$	1	-3	9
$X_4=$	1	1	1

7) If the primal problem has a non-unique optimal solution what can be said about the optimal solutions of the dual problem?

8) Prove that the rules (21), (22-a,b,c) of the dual method of solution of the primal problem are actually the same rules of the method of sections 2,3,4 for choosing the pivot element, if the dual problem is solved using this method.

9)

a) Construct the dual problem to the linear program of the example solved in section 5.

b) Solve this problem in a sequence of dual tables in the form (9-b)

c) Using <u>only</u> the inequalities of the dual problem as obtained in (a) above, and the optimal solution (2) of the primal problem, determine <u>directly</u> the optimal solution of the dual problem by using the theorm 4 and the relation (20) which have been proved in section 9.

12. The computer program

The general sequence of the different operations in the algorithm is given by the following block diagram. The details of each part is given in the flow chart. The FORTRAN program of the algorithm (at the end of this section) has been tested on the 1620 in the Operations Research Center of the Institute of National Planning.

The following symbols have are used in the flow chart and in the FORTRAN program; they have the following meaning:

M, N : the number of inequalities and the number of independent variables respectively.

M1 : M1 = M+1 = the total number of rows in the table.

N1 : N1 = N+1 = the total number of columns in the table.

MN1 : MN1 = M+N+1 = the total number of variables (including the objective function X_1).

A(I,J) : is the element of the table 2, which lies in the row I and in the column J.
The row number 1 is the row of the objective function, and the column number N1 is the column of the constant terms.

IXROW(I): Indicates the variable which is written in the row I.

IXCOL(J): Indicates the variable which is written in the column J.
At the beginning of the algorithm we have the table 2, so that the values of

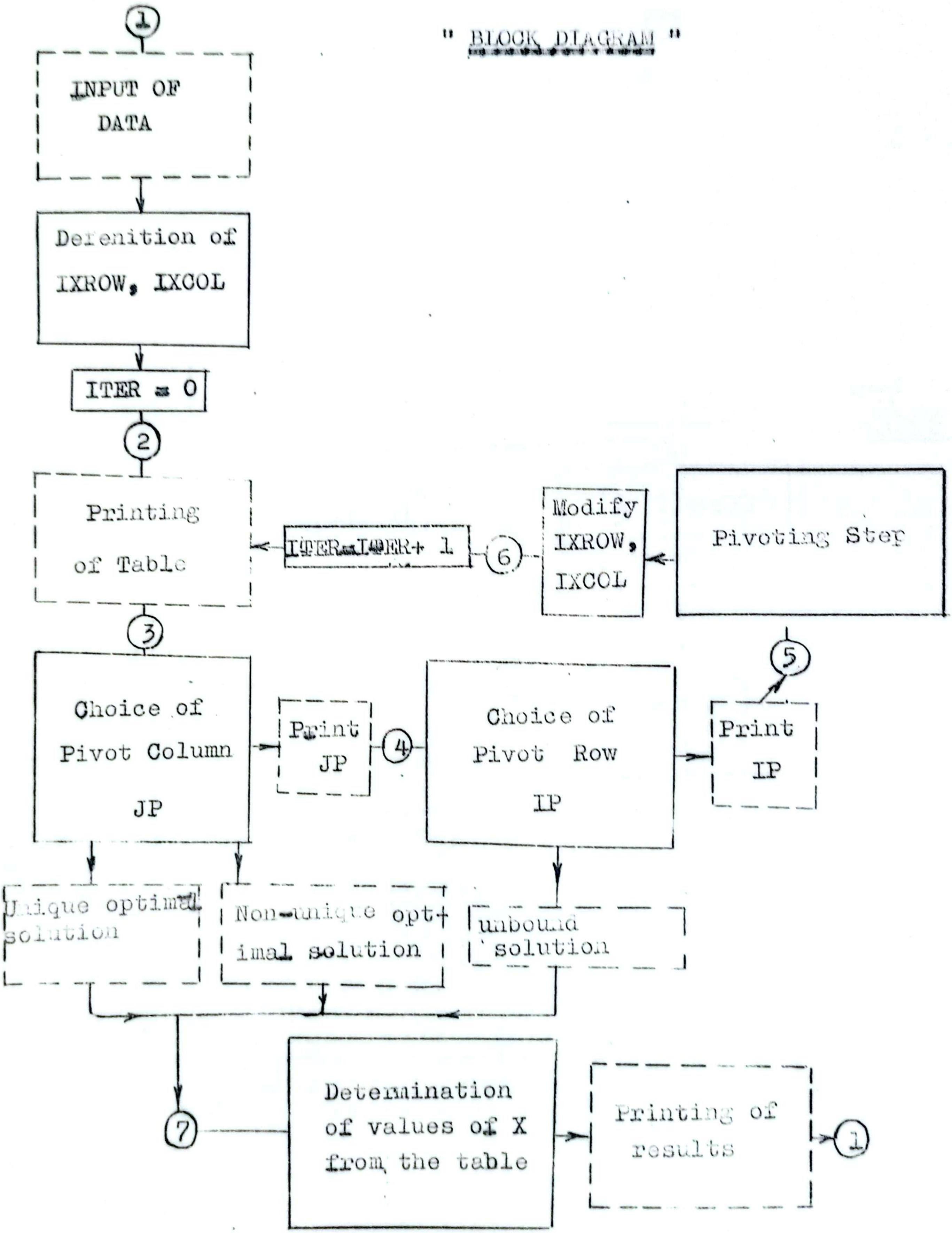
IXROW(I) = I (i.e. it represents the variable X_I)
for I = 1,2, ..., M1

and

IXCOL(J) = M1+J (i.e. it represents the variable X_{M1+J})

ITER	:	The number of iterations (i.e. the number of the pivoting operations) performed.
IP	:	The pivot row
JP	:	The pivot column
Q	:	The smallest absolute value of the charachteristic quotients
X(I)	:	The value of the variable X_i in the optimal solution; the optimal value of the objective function will be thus X_1 = X(1).

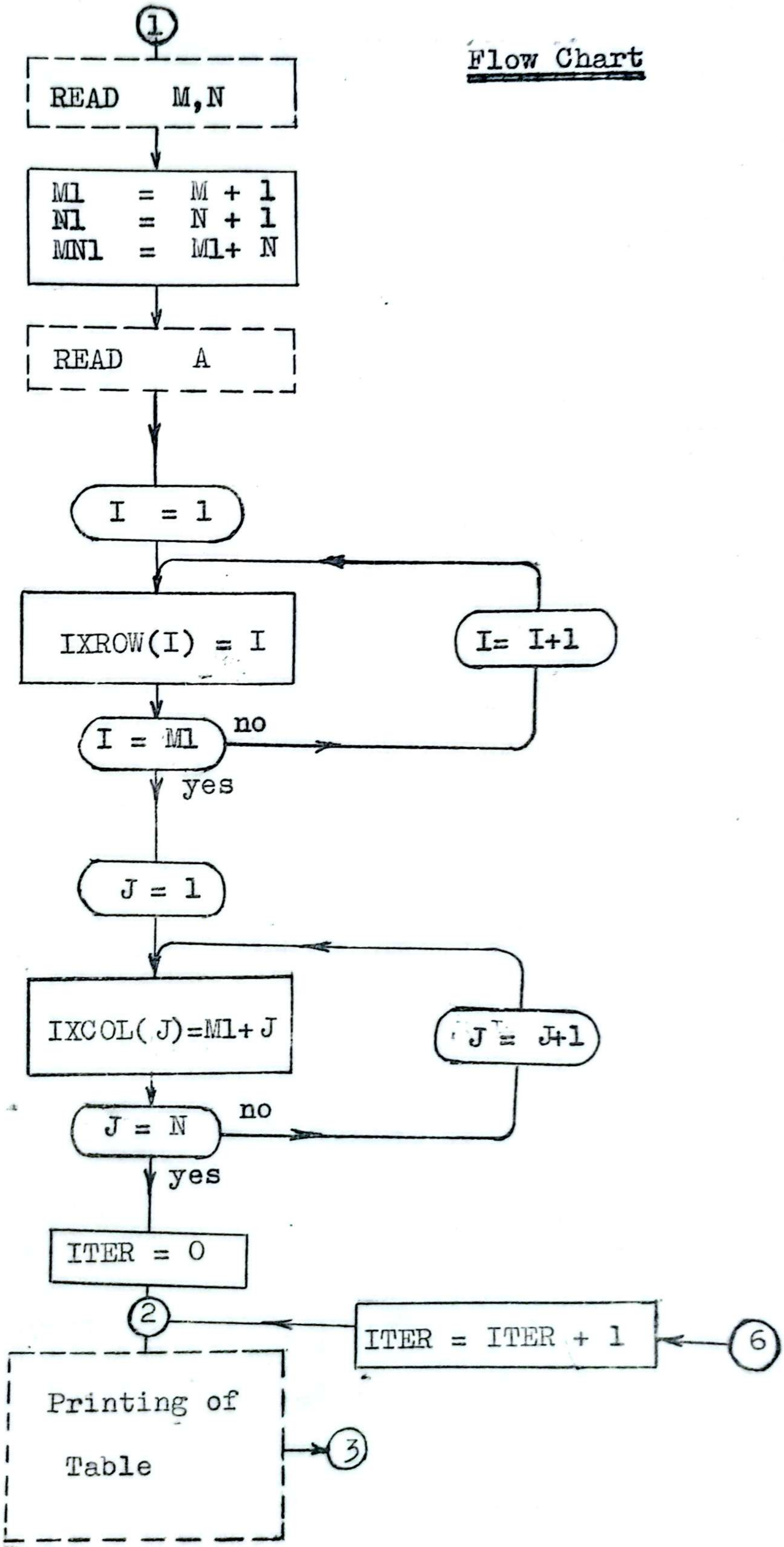
" BLOCK DIAGRAM "

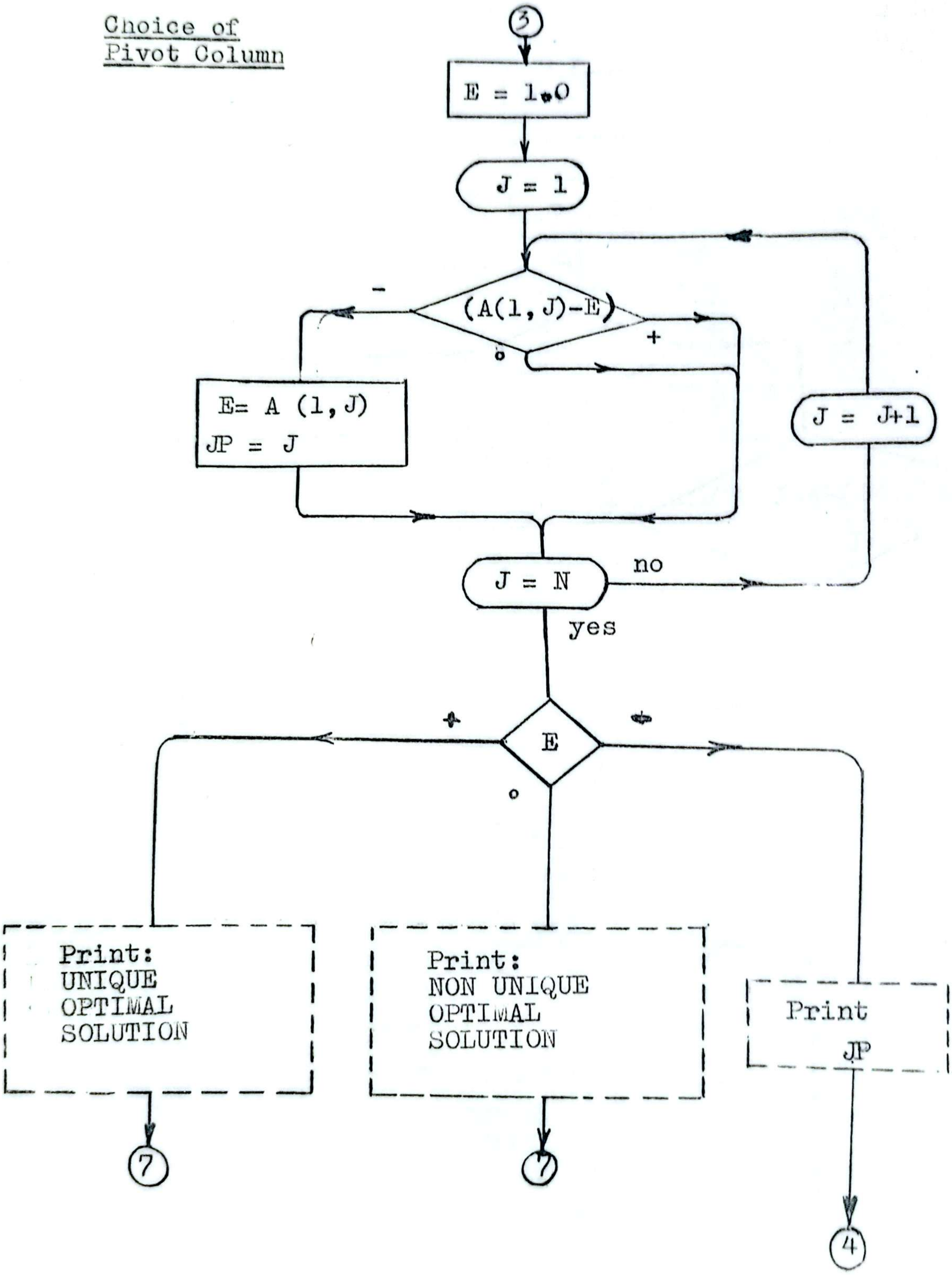


Flow Chart

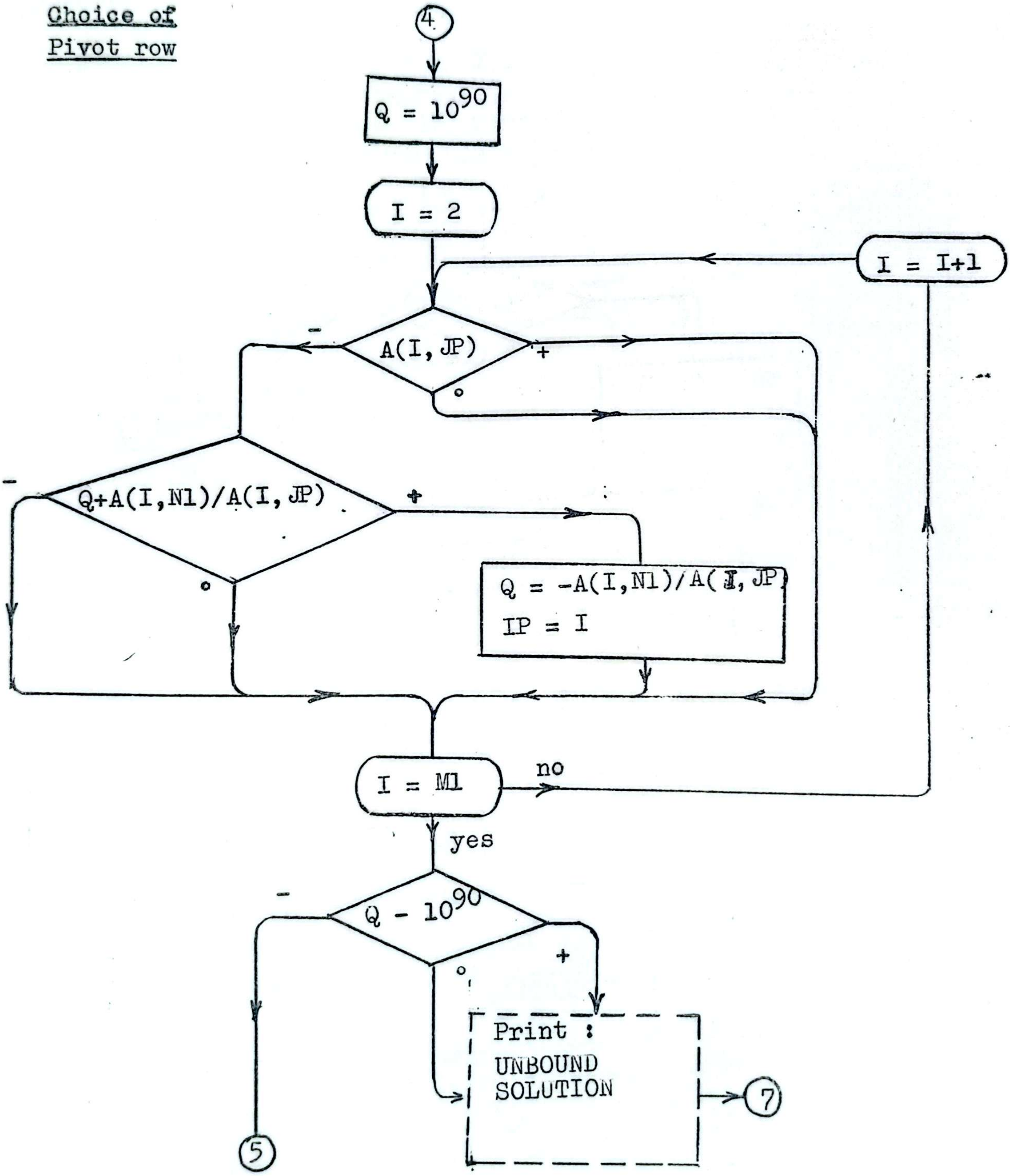
INPUT OF DATA

Definition of IXROW and IXCOL

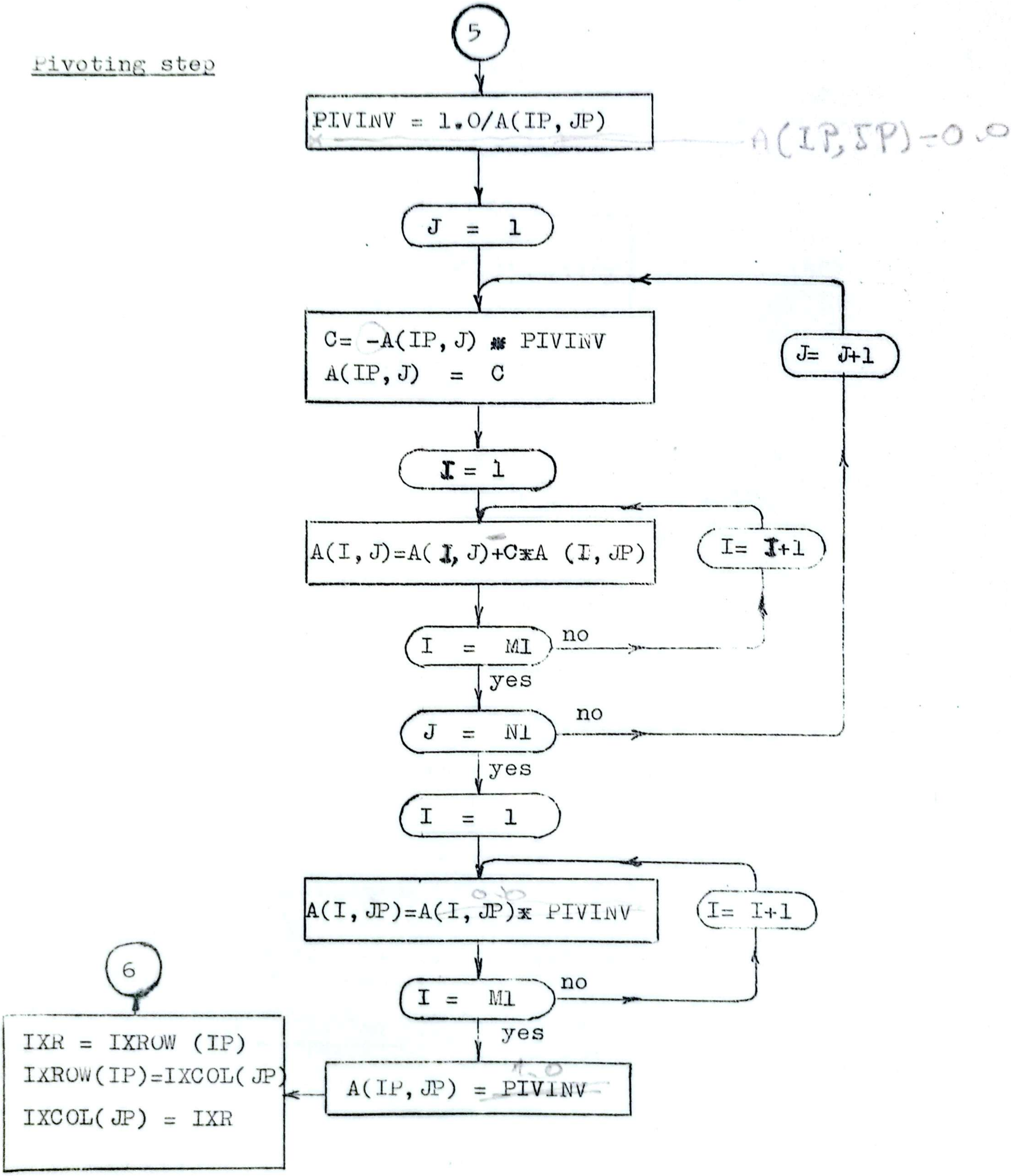



Choice of
Pivot Column
3
E = 1.0
J = 1
(A(1,J)-E)
-
+
0
E= A (1,J)
JP = J
J = J+1
J = N
no
yes
E
+
-
0
Print:
UNIQUE
OPTIMAL
SOLUTION
Print:
NON UNIQUE
OPTIMAL
SOLUTION
Print
JP
7
7
4

Choice of Pivot row

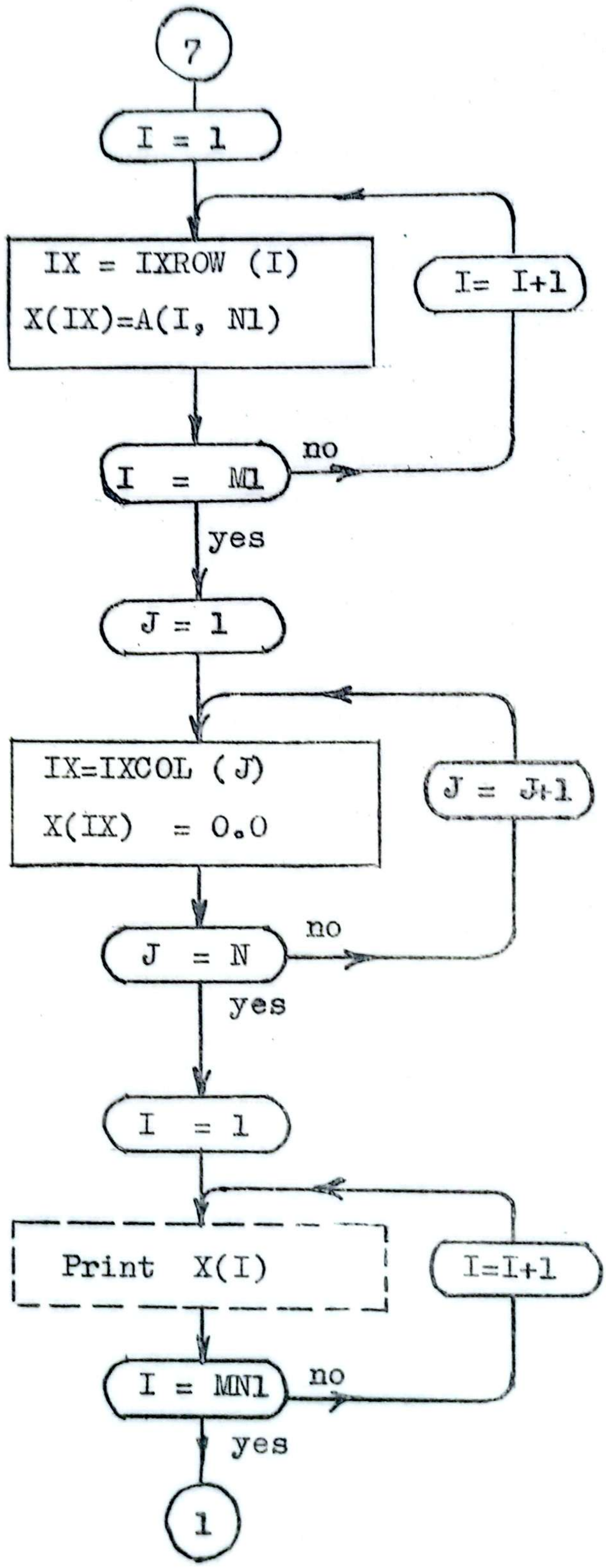


Pivoting step



Program termination: Definition of X

Printing of X



THE FORTRAN PROGRAM

```
C       SIMPLEX ALGORITHM
C       COMPACT FORM
        DIMENSION A(20,20), IXROW (20), IXCOL (20) , X (40)
  100   READ  1, M, N,NPROB
    1   FORMAT (2I3, I5)
        M1 = M + 1
        N1 = N + 1
        MN1= M1+ N
        DO2I = 1, M1
    2   READ 3, (A(I,J) , J=1, N1)
    3   FORMAT (8F10.4)
        DO 4 I = 1, M1
    4   IXROW (I) = I
        DO 5 J=1 , N
    5   IXCOL ( J) = M1 + J
        PRINT 6, NPROB, M, N
    6   FORMAT(///////I5//12HGIVEN TABLE / 3HM = ,I2, 3H N=,I2)
        ITER = 0
C       PRINTING OF TABLE
   10   PRINT 11, ITER
   11   FORMAT (////17H ITERATION NUMBER, I4)
        PRINT 12, (IXCOL ( J) , J=1, N)
   12   FORMAT (12X, 1HX,I2, 1HX,I2,12X,1HX,I2,12X,1HX,I2)
        DO 13 I=1, M1
   13   PRINT 14, IXROW (I), (A(I,J), J=1,N1)
   14   FORMAT (2H X, I2,2H = ,(3X,E12.5,3X,E12.5,3X,E12.5,3X,E12.5))
```

```
C       CHOICE OF PIVOT COLUMN
        E = 1.0
        DO 21  J=1  , N
        IF (A(1,J) - E) 20,21,21
     20 E = A (1,J)
        JP = J
     21 CONTINUE
        IF ( E ) 22, 42,44
     22 PRINT 23,  JP
     23 FORMAT (20H PIVOT COLUMN   JP = ,I2)

C       CHOICE OF PIVOT ROW

        Q =  1.0E + 90
        DO 26 I=2, M1
        IF (A(I,JP) 24, 26, 26
     24 IF (Q+A (I,N1)/A(I,JP)) 26, 26, 25
     25 Q = -A (I,N1)/A(I,JP)
        IP = 1
     26 CONTINUE
        IF (Q-1. OE+90) 27, 40, 40,
     27 PRINT 28,  IP
     28 FORMAT (17H PIVOT ROW   IP = , I2 // )

     40 PRINT 41, ITER
     41 FORMAT (17H UNBOUND SOLUTION, I12)
        IX = IXCOL ( JP)
        PRINT 46, IX
        GO TO 47
     42 PRINT 43, ITER
     43 FORMAT (28H OPTIMAL SOLTUION NON-UNIQUE, I12)
        GO TO 47
     44 PRINT 45, ITER
     45 FORMAT (17H OPTIMAL SOLUTION,I12)
     46 FORMAT (2H X, I2, 11H = INFINITY)
        GO TO 47
```

```
C       PIVOT STEP
   30 PIVINV = 1.0/A(IP, JP)
      A(IP, JP) = 0.0
      DO 31  J=1, N1
      C = -A(IP, J) * PIVINV
      A (IP, J) = C
      DO 31  I=1, M1
      A (I, J) = A (I, J) + C*A (I, JP)
   31 CONTINUE
      DO 32 I = 1, M1
   32 A(I, JP)=A(I, JP)*PIVINV        0.0
      A(IP , JP) = PIVINV             1.0
      IXR = IXROW (IP)
      IXROW (IP) = IXCOL ( JP)
      IXCOL ( JP) = IXR
      ITER = I[illegible] + 1
      GO TO 10

C       PROGRAM TERMINATION

   47 DO 48 I = 1, M1
      IX = IXROW (I)
   48 X (IX) = A (I,N1)
      DO 49 J = 1,N
      IX = 1XCOL ( J)
   49 X (IX) = 0.0
      DO 50 I = 1, MN1
   50 PRINT 51, I, X(I)
   51 FORMAT (2H X, I2, 2H = , E15. 4)
      GO TO 100

   52 END
```

References.

1. Dantzig

" Linear programming and extensions "

2. E. Stiefel

" Einführung in die numerische Mathematik "