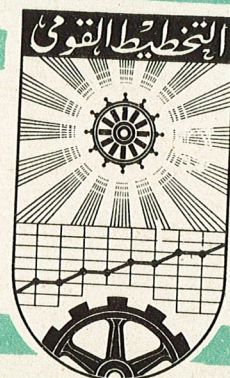


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Large Scale Linear Programming Problems

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CONTENTS

CHAPTER 1.	THE THEORETICAL FOUNDATIONS OF LINEAR PROGRAMMING	1
CHAPTER 2.	DIRECT METHODS	4
2.1.	The General Simplex Method	4
2.2.	The Revised Simplex Method	9
2.3.	Problems with Special Structure	11
2.4.	On the Method of the Effective Directions	16
2.5.	The Determination of the Effective Direc- tions	18
CHAPTER 3.	THE INDIRECT METHODS	27
3.1.	The Principle of Decomposition	27
3.2.	The Relaxation Procedure	32
CHAPTER 4.	THE PARAMETRIC PROGRAMMING	37
4.1.	The Reduced Problem	37
4.2.	The Extension of the Examination	
REFERENCES		46

INTRODUCTION

To-day the area of linear programming can be already regarded as theoretically resolved. In spite of this fact, serious difficulties may arise in connection with its practical application. The fundamental reason of these is the cumulation of the round-off errors arising during the execution of the calculations. The biasement caused by the calculation errors is increasing with the dimension of the problem.

The examination of this question is very important, because the solution of large scale problems is increasingly demanded in the practice.

The first chapter expounds the theoretical foundations of the linear programming.

Here the methods in question are divided into two groups, into the so-called direct and indirect methods. chapter 2 deals with the first group, while chapter 3 deals with the second one.

Among the direct methods the simplex method and the method of the effective directions are discussed. The different known variants of the method of the effective directions can be discussed on the basis of the same principle by giving a suitably chosen normalizing condition. The individual methods are distinguished according to the choice of this norma-

ization condition. So also such variants can be developed, which till now have not occurred in the literature.

Chapter 3, including the indirect methods, depends actually on two large problems. These are the decomposition principle due to DANTZIG and from WOLFE, the relaxation principle due to ROSEN. Here, only problems with special structure are considered, otherwise the application of these methods is not economical.

Chapter 4 deals with the parametric programming, but only that case is discussed, when the constants of the problem are depending on a single parameter, and they are only its linear functions.

Finally a few words about the notations. The matrices are denoted

by underlined capital letters, while the column vectors are denoted by

underlined small letters. The scalars, which here represent always real

numbers, are denoted by Latin or Greek letters; without underlining. Other

notations:

$\bar{A}^T$: is the transpose of the matrix A .

$\bar{E}$: is the identity matrix

$\bar{E}_n$: is the unit vector of order n .

$\bar{e}_i$: is the i th unit vector.

$\bar{1}$: is the sum vector, whose all components are 1.

E_n : is the real Euclidean space of dimension n .

Chapter 1.

The Theoretical Foundations of the Linear Programming:

As it is known, any linear programming problem can be written in the following form:

$$(1.1) \quad \begin{aligned} \underline{x} &\geq \underline{0} \\ \underline{Ax} &\leq \underline{b} \\ \underline{c}^T \underline{x} &\rightarrow \max \end{aligned}$$

or in the corresponding standard form:

$$\max \left\{ \underline{c}^T \underline{x} \mid \underline{x} \in L_p \right\},$$

where the set of the feasible programs is:

$$L_p = \left\{ \underline{x} \mid \underline{x} \geq \underline{0}, \underline{Ax} \leq \underline{b} \right\}.$$

If we introduce the slack variable $\hat{\underline{u}} = \underline{b} - \underline{Ax}$ then the problem (1.1) takes the following form:

$$(1.2) \quad \begin{aligned} \underline{x} &\geq \underline{0} & \hat{\underline{u}} &\geq \underline{0} \\ \underline{Ax} + \hat{\underline{u}} &= \underline{b} \\ \underline{c}^T \underline{x} + \underline{0}^T \hat{\underline{u}} &\rightarrow \max \end{aligned}$$

The dual problem of (1.1) takes the following form:

$$(1.3) \quad \begin{aligned} \underline{u} &\geq \underline{0} \\ \underline{u}^T \underline{A} &\geq \underline{c}^T \\ \underline{u}^T \underline{b} &\rightarrow \min. \end{aligned}$$

We can formulate it also in the form:

$$\min \left\{ \underline{u}^T \underline{b} \mid \underline{u} \in L_d \right\}$$

Where,

$$L_d = \{ \underline{u} \mid \underline{u} \geq \underline{0}, \quad \underline{u}^T \underline{A} \geq \underline{c}^T \}$$

is the set of the feasible solutions of the dual problem. By introducing the slack variable $\hat{\underline{x}}^T = \underline{u}^T \underline{A} - \underline{c}^T$, the dual problem (1.3) can be formulated in the form

$$(1.4) \quad \begin{aligned} \underline{u} &\geq \underline{0}, \quad \hat{\underline{x}} \geq \underline{0} \\ \underline{u}^T \underline{A} - \hat{\underline{x}}^T &= \underline{c}^T \\ \underline{u}^T \underline{b} + \underline{0}^T \hat{\underline{x}} &\rightarrow \min \end{aligned}$$

If there exists such a program $\underline{x}_0 \in L_p$, or resp. $\underline{u}_0 \in L_d$, that for any $\underline{x} \in L_p$, resp. $\underline{u} \in L_d$ the inequalities $\underline{c}^T \underline{x} \leq \underline{c}^T \underline{x}_0$, resp. $\underline{u}^T \underline{b} \geq \underline{u}_0^T \underline{b}$ are valid, then we call the vector $\underline{x}_0$ resp. the vector $\underline{u}_0$ the optimal solution.

Now we turn our attention to the discussion of the fundamental theorems of duality.

Theorem 1.1: (Symmetricity)

The dual of the dual problem is equivalent to the primal problem, and conversely.

Theorem 1.2: (Weak duality)

$$\sup \{ \underline{c}^T \underline{x} \mid \underline{x} \in L_p \} \leq \inf \{ \underline{u}^T \underline{b} \mid \underline{u} \in L_d \}$$

Theorem 1.3: (Strong duality)

The program $\underline{x}_0 \in L_p$ is an optimal solution, if and only if there exists a program $\underline{u}_0 \in L_d$, for

which,

$$\underline{c}^T \underline{x}_0 = \underline{u}_0^T \underline{b}$$

It follows from theorem 1.1, that the $\underline{u}_0$ of theorem 1.3 represents an optimal solution of the dual problem. If the pair of vector $(\underline{x}_0, \underline{u}_0)$ is the optimal solution for the two dual problems, then the slack variables are the following:

$$\hat{\underline{u}}_0 = \underline{b} - \underline{A}\underline{x}_0$$

$$\hat{\underline{x}}_0^T = \underline{u}_0^T \underline{A} - \underline{c}^T$$

By means of these we can formulate

Theorem 1.4: (Complementarity)

$$\hat{\underline{x}}_0^T \underline{x}_0 = \underline{u}_0^T \hat{\underline{u}}_0 = 0$$

Theorem 1.5: (Existence)

If both the sets L_p and L_d are non-empty, then the set of the optimal points is also non-empty.

Theorem 1.6:

The set of the optimal points is such a convex polyhedral, whose extreme points are equivalent to the optimal extreme points of L_p .

Chapter 2

Direct Methods

2.1. The General Simplex Method:

The best known method for solving the linear programming problems is the simplex method. This is a method based on the elementary basis transformation. Namely, applying the simplex method to solve a linear programming problem is as much as looking for such a basis over a sequence of elementary basis transformations, which directly gives the solution of the problem. The way of the realization will be discussed taken. Now we make some fundamental remarks in connection with the set L_p .

As it is known, under the conditions formulated in chapter 1 the set L_p is a convex polyhedral, whose points are the feasible solutions of the inequality system

$$\underline{x} \geq 0$$

$$\underline{Ax} \leq \underline{b}$$

By introducing the slack variable $\hat{u}$, it can be written also in the complete form, as in (1.2)

$$\underline{x} \geq 0 \quad \hat{u} \geq 0$$

$$\underline{Ax} + \hat{u} = \underline{b}$$

we associate with this the following coefficient matrix

$$[\underline{A}, \underline{E}]$$

Relying upon these we shall introduce the concept of the basic solution, which is of fundamental importance. The solution $\underline{x}_0 \in L_p$ is said to be a basic solution, if the coefficient vectors belonging to the positive com-

ponents of the vector-pair $(\underline{x}_0, \hat{\underline{u}}_0 = \underline{b} - \underline{A} \underline{x}_0)$ are linearly independent.

Relying on the new concept we formulate some important theorems.

Theorem 2.1:

If $L_p \neq \emptyset$, then there exists a basic solution, too.

Theorem 2.2:

To every basic solution of L_p corresponds an extreme point, and to every extreme point of L_p corresponds a basic solution.

Theorem 2.3:

The number of the basic solutions, i.e. the number of the extreme points is finite.

Now, relying on the foregoing we can already formulate the basic idea of the simplex method. We choose one of the extreme points of L_p , then from this we go to a better neighbouring extreme point with the help of one step of elementary basis transformation. Then we repeat this procedure as many times as necessary.

It follows from the previous theorems that under the given conditions we obtain an optimal extreme point in a finite number of steps. This extreme point according to theorem (1.6), represents an optimal solution for problem (1.1).

We shall associate the following tableau with the pair of dual problems of (1.1) and (1.3).

(2.1.1)

	$\underline{x}^T$	
$\underline{U}$	$\underline{A}$	$\underline{b}$
	$\underline{c}^T$	$\underline{o}$

We call this tableau the simplex tableau and regard it as the representation of the following relations.

$$(2.1.2) \quad \hat{\underline{U}} + \underline{A}\underline{x} = \underline{b}$$

$$(2.1.3) \quad -\hat{\underline{x}}^T + \underline{u}^T \underline{A} = \underline{c}^T$$

As known, the components of $\underline{u}$ represent the basic variables, and the components of $\underline{x}$ the non-basic variables with respect to the primal problem, while the reversed place is due to the dual problem. So, from the two relations (2.1.2) and (2.1.3) we obtain the pseudo-basic solutions, by writing 0 in place of the non-basic variables.

Let us consider the partitioning of the matrix $\underline{A}$

$$\underline{A} = \begin{bmatrix} \underline{A}_{11} & \underline{A}_{12} \\ \underline{A}_{21} & \underline{A}_{22} \end{bmatrix}$$

Where $\underline{A}_{11}$ is a non-singular block of order r . Applying this partition we can write the tableau (2.1.1) as:

(2.1.4)

	$\underline{x}_1^T$	$\underline{x}_2^T$	
$\underline{u}_1$	$\underline{A}_{11}$	$\underline{A}_{12}$	$\underline{b}_1$
$\underline{u}_2$	$\underline{A}_{21}$	$\underline{A}_{22}$	$\underline{b}_2$
	$\underline{c}_1^T$	$\underline{c}_2^T$	0

then by performing the corresponding basis transformation we obtain the tableau

$$(2.1.5) \quad \begin{array}{c|cc|c} & \hat{u}_1^T & x_2^T & \\ \hline x_1 & A_{11}^{-1} & A_{11}^{-1} A_{12} & A_{11}^{-1} b_1 \\ u_2 & -A_{21} A_{11}^{-1} & A_{22} - A_{21} A_{11}^{-1} A_{12} & b_2 - A_{21} A_{11}^{-1} b_1 \\ \hline & -C_1^T A_{11}^{-1} & C_2^T - C_1^T A_{11}^{-1} A_{12} & -C_1^T A_{11}^{-1} b_1 \end{array}$$

This tableau represents the formulas:

$$(2.1.6) \quad x_1 + A_{11}^{-1} \hat{u}_1 + A_{11}^{-1} A_{12} x_2 = A_{11}^{-1} b_1$$

$$\hat{u}_2 - A_{21} A_{11}^{-1} \hat{u}_1 + (A_{22} - A_{21} A_{11}^{-1} A_{12}) x_2 = b_2 - A_{21} A_{11}^{-1} b_1$$

with respect to the primal problem, and the formulas:

$$(2.1.7) \quad -\hat{u}_1^T + \hat{x}_1^T A_{11}^{-1} - u_2^T A_{21} A_{11}^{-1} = -C_1^T A_{11}^{-1}$$

$$-\hat{x}_2^T + \hat{x}_1^T A_{11}^{-1} A_{12} + u_2^T (A_{22} - A_{21} A_{11}^{-1} A_{12}) = C_2^T - C_1^T A_{11}^{-1} A_{12}$$

with respect to the dual problem.

In the tableau (2.1.5), the basic variables of the primal problem are represented by the components of (x_1, u_2) , and the basic variables of the dual problem by the components of (u_1^T, x_2^T) . Accordingly the pseudo-basic solutions take the form:

$$(2.1.8) \quad \underline{x} = \begin{bmatrix} A_{11}^{-1} b_1 \\ 0 \end{bmatrix}, \quad \hat{\underline{u}} = \begin{bmatrix} 0 \\ b_2 - A_{21} A_{11}^{-1} b_1 \end{bmatrix}$$

$$(2.1.9) \quad \underline{u}^T = [C_1^T A_{11}^{-1}, 0^T], \quad \hat{\underline{x}}^T = [0^T, C_2^T + C_1^T A_{11}^{-1} A_{12}]$$

The value of the objective function for both pseudo-basic solutions is

$$\underline{c}_1^T \underline{A}_{11}^{-1} \underline{b}_1.$$

If in the last column of the tableau (2.1.5) there is at least one 0 among the components of the vector

$$\begin{bmatrix} \underline{A}_{11}^{-1} \underline{b}_1 \\ \underline{b}_2 - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{b}_1 \end{bmatrix}$$

then we say, that degeneracy occurred in the primal problem. And if 0 can be found among the components of the vector $\left[-\underline{c}_1^T \underline{A}_{11}^{-1}, \underline{c}_2^T - \underline{c}_1^T \underline{A}_{11}^{-1} \underline{A}_{12} \right]$, then we speak about the degeneracy of the dual problem. In this case the corresponding pseudo-basic solutions are considered to be degenerate solutions. If a simplex tableau gives such a pseudo-basic solution with respect to the primal problem, which is at the same time a feasible solution, too, then we call this simplex tableau a primal feasible tableau. The dual feasible tableau can be defined in a similar way.

Theorem 2.4:

If the simplex tableau is both primal and dual feasible, then the tableau gives an optimal solution.

Two basic variants of the simplex method are distinguished: the primal simplex method and the dual simplex method. The basic idea for the first is, that we approach the optimum through a sequence of primal feasible tableaus, and for the second this approach is made through a sequence of dual feasible tableaus.

2.2. The Revised Simplex Method:

To present this method first from the constants of the problem (1.1) resp. (1.2) we shall construct the matrix:

$$(2.2.1) \quad \underline{L}_0 = \begin{bmatrix} \underline{A} & \underline{E} & \underline{b} \\ \underline{C}^T & \underline{0} & 0 \end{bmatrix}$$

Which is the matrix of the initial data, to which belongs the basis determined by the matrix:

$$\underline{B}_0 = \begin{bmatrix} \underline{E} & \underline{0} \\ \underline{0}^T & 1 \end{bmatrix}$$

The simplex method can be characterized in such a way, that we exchange the column vectors of the block $\begin{bmatrix} \underline{E} \\ \underline{0}^T \end{bmatrix}$ in the basis by suitably chosen column vectors of the block $\begin{bmatrix} \underline{A} \\ \underline{C}^T \end{bmatrix}$ of the matrix $\underline{L}_0$.

Let us assume, that at the s th step, $\underline{B}_0$ takes the form:

$$\underline{B}_s = \begin{bmatrix} \underline{D} & \underline{0} \\ \underline{d}^T & 1 \end{bmatrix}$$

In this basis $\underline{L}_0$ becomes:

$$\underline{L}_s = \underline{B}_s^{-1} \underline{L}_0$$

$\underline{L}_s$ represents the s th simplex tableau. From the definition of $\underline{L}_0$ and $\underline{B}_s$ we can give $\underline{L}_s$ in a more detailed form:

$$(2.2.2) \quad \underline{L}_s = \begin{bmatrix} \underline{D}^{-1} \underline{A} & \underline{D}^{-1} & \underline{D}^{-1} \underline{b} \\ \underline{C}^T - \underline{d}^T \underline{D}^{-1} \underline{A} & -\underline{d}^T \underline{D}^{-1} & -\underline{d}^T \underline{D}^{-1} \underline{b} \end{bmatrix}$$

Relying on this result, it can be seen, that any element of $\underline{L}_S$ can be calculated easily from the initial data, if we know the matrix $\underline{D}^{-1}$. Namely, knowing $\underline{D}^{-1}$ we can perform the s th simplex iteration in such a way, that we do not determine every element of the matrix $\underline{L}_S$, only those which are needed.

In the case of the primal simplex method we can arrange this in the following way:

1. We determine the vector $\underline{D}^{-1} \underline{b}$.
2. We determine the vector $-\underline{d}^T \underline{D}^{-1}$. If $-\underline{d}^T \underline{D}^{-1} \leq \underline{0}^T$, we turn to step 4.
3. We choose the largest positive element of the vector $-\underline{d}^T \underline{D}^{-1}$. Let it be the k th element. Then we choose the pivot element in the column vector $\underline{D}^{-1} \underline{e}_k$ and turn to the next iteration.
4. On the basis of the formula $(\underline{C}^T - \underline{d}^T \underline{D}^{-1} \underline{A}) \underline{e}_i$ we determine one after the other the elements of this vector, but we stop at the first positive element. Let it be the k th element. Then we determine the vector $\underline{D}^{-1} \underline{A} \underline{e}_k$, from its components we choose the pivot element, then we turn to the next iteration.
5. If $\underline{C}^T - \underline{d}^T \underline{D}^{-1} \underline{A} \leq \underline{0}^T$, then $\underline{L}_S$ represents an optimal tableau.

In the case of the dual simplex method we proceed according to the following:

1. We determine the vector $[\underline{C}^T - \underline{d}^T \underline{D}^{-1} \underline{A}, -\underline{d}^T \underline{D}^{-1}]$.
2. We determine successively the components of the vector $\underline{D}^{-1} \underline{b}$, but we stop at the first negative component. Let it be the j th component. We determine the row vector $[\underline{e}_j^T \underline{D}^{-1} \underline{A}, \underline{e}_j^T \underline{D}^{-1}]$,

choose the pivot element from its components, then we go to the next iteration.

3. If $\underline{D}^{-1} \underline{b} \geq 0$, then $\underline{L}_S$ represents an optimal tableau. As it can be seen, in the way described above really only a small part of the elements of $\underline{L}_S$ has to be determined.

This is an important circumstance from the point of view of occupying the storage capacity. From (2.2.2.) it can be seen, that we obtain the data to be determined directly from the initial data. This fact can help in decreasing the round-off errors. Here, a serious difficulty arises. Namely, in some way the inverse of $\underline{D}$ must be determined. Of course it could be determined by the help of the general simplex method, but this would hardly be helpful in our problem caused by the round-off errors. According to the experience, however, it is possible to achieve considerable improvement, if we apply the so-called product form for the determination of the inverse.

2.3. Problems with Special Structure:

The methods discussed before apply generally to the solution of any linear programming problem. Yet the special structure of the given problem may involve certain possibilities for simplification. In the following we want to exploit this possibility in connection with the following problem:

$$\begin{aligned}
 & \underline{x}_0 \geq 0, \quad \underline{x}_1 \geq 0, \dots, \underline{x}_p \geq 0 \\
 (2.3.1.) \quad & \underline{K}_0 \underline{x}_0 + \underline{K}_1 \underline{x}_1 + \dots + \underline{K}_p \underline{x}_p \leq \underline{b}_0 \\
 & \underline{A}_1 \underline{x}_1 \leq \underline{b}_1 \\
 & \underline{A}_{p-p} \underline{x}_p \leq \underline{b}_p \\
 & \underline{C}_0^T \underline{x}_0 + \underline{C}_1^T \underline{x}_1 + \dots + \underline{C}_p^T \underline{x}_p \longrightarrow \max
 \end{aligned}$$

A possible economic interpretation of this problem makes easier the understanding of the sequence of ideas to be applied. Let us assume, that the centre of a certain economic unit controls the work of p different sectors. Besides it has to be taken into consideration that the centre has only the resources given by the vector $\underline{b}_0$ at its disposal, while the resources determined by the vectors $\underline{b}_1, \dots, \underline{b}_p$ fall within the competence of the respective sectors. The matrices $\underline{K}_0, \underline{K}_1, \dots, \underline{K}_p$ contain the technical coefficients for the central resources, which the matrices $\underline{A}_1, \dots, \underline{A}_p$ are the technical coefficients for the sector resources. The components of the vectors $\underline{C}_0, \underline{C}_1, \dots, \underline{C}_p$ show the per unit efficiency of the central, resp. sector activities. According to this, $\underline{x}_0$ gives information on the level of the central activities, while $\underline{x}_1, \dots, \underline{x}_p$ on that of the sector activities. If the number of the components of the vectors $\underline{b}_0, \underline{b}_1, \dots, \underline{b}_p$ are the scalars $m_0, m_1, \dots, m_p$, then the number of the central constraints is m_0 , and the number of the sector constraints is $m_1 + \dots + m_p = m$. The complete coefficients matrix belonging to the problem (2.3.1) takes the form.

$$(2.3.2) \quad \left[\begin{array}{cccc|c} \underline{K}_0 & \underline{K}_1 & \dots & \underline{K}_p & \underline{E}_{m_0} \\ & \underline{A}_1 & & & \underline{E}_{m_1} \\ & & \ddots & & \vdots \\ & & & \underline{A}_p & \underline{E}_{m_p} \end{array} \right] = [\underline{A}, \underline{E}]$$

Where the symbols $\underline{E}_{m_0}, \underline{E}_{m_1}, \dots, \underline{E}_{m_p}$ represent identity matrices of order

$m_0, m_1, \dots, m_p$.

So the complete problem can be written in the form:

$$\begin{aligned}
 & \underline{x}_0 \geq 0, \underline{x}_1 \geq 0, \dots, \underline{x}_p \geq 0, \hat{\underline{u}}_0 \geq 0, \hat{\underline{u}}_1 \geq 0, \dots, \hat{\underline{u}}_p \geq 0 \\
 (2.3.3) \quad & \underline{K}_0 \underline{x}_0 + \underline{K}_1 \underline{x}_1 + \dots + \underline{K}_p \underline{x}_p + \hat{\underline{u}}_0 = \underline{b}_0 \\
 & \underline{A}_1 \underline{x}_1 + \hat{\underline{u}}_1 = \underline{b}_1 \\
 & \underline{A}_p \underline{x}_p + \hat{\underline{u}}_p = \underline{b}_p \\
 & \underline{C}_0^T \underline{x}_0 + \underline{C}_1^T \underline{x}_1 + \dots + \underline{C}_p^T \underline{x}_p + \underline{0}^T \hat{\underline{u}}_0 + \underline{0}^T \hat{\underline{u}}_1 + \dots + \underline{0}^T \hat{\underline{u}}_p \rightarrow \max
 \end{aligned}$$

In the solution of the problem we shall proceed according to the revised simplex method. According to the arguments set forth in point 2.2, the determination of the inverse of the matrix $\underline{D}$. The special structure of the problem now to be considered provides certain possibilities for simplification. To exploit these possibilities we can rely upon the ideas applied by DANTZIG to develop the so-called generalized upper bounding (GUB) technique.

The column vectors of $\underline{D}$ are chosen from among the column vectors of the coefficient matrix (2.3.2.). Accordingly-by a possible rearrangement of the columns-the following construction can be given for the matrix $\underline{D}$ in question:

$$(2.3.4) \quad \underline{D} = \left[\begin{array}{ccc|ccc} \underline{K}_{0s} & \underline{K}_{1s} & \dots & \underline{K}_{ps} & \underline{K}_{1h} & \dots & \underline{K}_{ph} \\ & \underline{A}_{1s} & & & \underline{A}_{1h} & & \\ & & \ddots & & & \ddots & \\ & & & \underline{A}_{ps} & & & \underline{A}_{ph} \end{array} \right]$$

Here the column vectors of $\underline{K}_{0s}$ are chosen from the column vectors of $[\underline{K}_0, \underline{E}_{m_0}]$

, while the column vectors of the blocks,

$$\begin{bmatrix} \underline{K}_{is} \\ \vdots \\ \underline{A}_{is} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \underline{K}_{ih} \\ \vdots \\ \underline{A}_{ih} \end{bmatrix} \quad (i=1,2,\dots,p)$$

are from the column vectors of the block

$$\begin{bmatrix} \underline{K}_i & \vdots & \underline{0} \\ \vdots & \ddots & \vdots \\ \underline{A}_i & \vdots & \underline{E}_{mi} \end{bmatrix} \quad (i=1,2,\dots,p)$$

The matrix $\underline{D}$ can be written also in the more concise form:

$$(2.3.5) \quad \underline{D} = \left[\begin{array}{cc|c} \underline{K}_{os} & \underline{K}_s & \underline{K}_h \\ \hline \underline{0} & \underline{A}_s & \underline{A}_h \end{array} \right]$$

Where

$$\underline{K}_s = [\underline{K}_{1s}, \dots, \underline{K}_{ps}]$$

$$\underline{K}_h = [\underline{K}_{1h}, \dots, \underline{K}_{ph}]$$

$$\underline{A}_s = \begin{bmatrix} \underline{A}_{1s} & & \\ & \ddots & \\ & & \underline{A}_{ps} \end{bmatrix}$$

and

$$\underline{A}_h = \begin{bmatrix} \underline{A}_{1h} & & \\ & \ddots & \\ & & \underline{A}_{ph} \end{bmatrix}$$

Moreover, if we use also the symbols

$$\left[\underline{K}_{os}, \underline{K}_s \right] = \hat{\underline{K}}_s$$

and

$$\left[\underline{0}, \underline{A}_s \right] = \hat{\underline{A}}_s$$

then we obtain the formula

$$(2.3.6) \quad \underline{D} = \begin{bmatrix} \hat{\underline{K}}_{\underline{s}} & \underline{K}_{\underline{h}} \\ \hat{\underline{A}}_{\underline{s}} & \underline{A}_{\underline{h}} \end{bmatrix}$$

We shall call the variables belonging to the column vectors of the block $\begin{bmatrix} \hat{\underline{K}}_{\underline{s}} \\ \hat{\underline{A}}_{\underline{s}} \end{bmatrix}$ the key basic variables, while the variables belonging to the block $\begin{bmatrix} \underline{K}_{\underline{h}} \\ \underline{A}_{\underline{h}} \end{bmatrix}$ the auxiliary basic variables. From the foregoing follows, that such a partition of the basic variables is always realizable in the given problem. The following statements are obvious, too:

- Every auxiliary basic variable is a sector variable.
- Beside the central variables there may be also sector variables among the key basic variables.
- The number of the key basic variables is m_0 , while the number of the auxiliary basic variables is $m_1 + \dots + m_p$.

The inverse of $\underline{D}$ is

$$(2.3.7) \quad \underline{D}^{-1} = \begin{bmatrix} \underline{K}^{-1} & -\underline{K}^{-1} \underline{K}_{\underline{h}} \underline{A}_{\underline{h}}^{-1} \\ -\underline{A}_{\underline{h}}^{-1} \hat{\underline{A}}_{\underline{s}} \underline{K}^{-1} & \underline{A}_{\underline{h}}^{-1} (\underline{E}_m + \hat{\underline{A}}_{\underline{s}} \underline{K}^{-1} \underline{K}_{\underline{h}} \underline{A}_{\underline{h}}^{-1}) \end{bmatrix}$$

Where

$$(2.3.8) \quad \underline{K} = \hat{\underline{K}}_{\underline{s}} - \underline{K}_{\underline{h}} \underline{A}_{\underline{h}}^{-1} \hat{\underline{A}}_{\underline{s}}$$

By knowing $\underline{A}_{\underline{h}}^{-1}$ and $\underline{K}^{-1}$, it is really simple to determine the inverse matrix $\underline{D}^{-1}$, which takes a prominent part in the arrangement of the revised simplex method. Notice has to be taken of only that in every iteration step the key basic variables must be separated from the auxiliary basic variables. First the auxiliary basic variables have to be marked, then the key basic variables. The auxiliary basic variables are marked out simply in such a way, that from among the sector variables, we choose $m_1 + \dots + m_p$ basic variables. The remaining basic variables are the key basic variables.

2.4. On the Method of the Effective Directions:

We can formulate the problem (1.1) in the form:

$$(2.4.1) \quad \underline{A}' \underline{x} \leq \underline{b}'$$

$$\underline{C}^T \underline{x} \rightarrow \max$$

where $\underline{A}' = \begin{bmatrix} -\underline{E} \\ \underline{A} \end{bmatrix}$ and $\underline{b}' = \begin{bmatrix} \underline{0} \\ \underline{b} \end{bmatrix}$

According to these,

$$L_p = \{ \underline{x} \mid \underline{A}' \underline{x} \leq \underline{b}' \}$$

We define the set of the feasible directions belonging to point

$$\underline{x}_0 \in L_p:$$

$$R = \{ \underline{r} \mid \underline{r} = \underline{x} - \underline{x}_0, \underline{x} \in L_p, \underline{x} \neq \underline{x}_0 \}$$

We normalize the vector $\underline{r} \in R$ in a sense. We mention three special cases:

$$a) \quad |\underline{r}| = 1$$

($\underline{Q}$ is non-singular)

$$b) \quad |\underline{Q} \underline{r}| = 1$$

$$c) \quad -1 \leq \underline{r} \leq 1$$

The program $\underline{x}_0 \in L_p$ satisfies the constraints of (2.4.1) partly in

the form of equalities and partly as strict inequalities. So we can apply

such partition

$$\underline{A}' = \begin{bmatrix} \underline{A}_1 \\ \underline{A}_2 \end{bmatrix}, \text{ resp. } \underline{b}' = \begin{bmatrix} \underline{b}_1 \\ \underline{b}_2 \end{bmatrix}$$

for $\underline{A}'$ and $\underline{b}'$, where

$$(2.4.2) \quad \underline{A}_1 \underline{x}_0 = \underline{b}_1$$

and

$$(2.4.3) \quad \underline{A}_2 \underline{x}_0 < \underline{b}_2$$

We shall say, that for $\underline{x}_0$ the (2.4.2) represents the active restrictions, and (2.4.3) the non-active restrictions.

Theorem 2.5:

The set of the feasible directions belonging to $\underline{x}_0$ is equivalent to the set of the ~~non-trivial~~ solutions of the inequality system

$$\underline{A}_1 \underline{r} \leq \underline{0}$$

We shall define the set of the effective directions belonging to $\underline{x}_0$ in the following way.

$$(2.4.4.) \quad \hat{R} = \left\{ \underline{r} \mid \underline{r} \in R, \quad \underline{C}^T \underline{r} > 0 \right\}$$

An important property of the effective directions is expressed by:

Theorem 2.6:

If $\underline{r}_0 \in \hat{R}$, then there is a scalar $\lambda_0 > 0$, such that for any $0 < \lambda \leq \lambda_0$

there hold

$$\underline{x}_\lambda = \underline{x}_0 + \lambda \underline{r}_0 \in L_p$$

and

$$\underline{C}^T \underline{x}_0 < \underline{C}^T \underline{x}_\lambda$$

A point $\underline{x}_s \in L_p$, to which does not belong any effective direction, is called a stationary point.

Theorem 2.7:

If $\underline{x}_s \in L_p$ is a stationary point, then it is optimal at the same time.

2.5. The Determination of the Effective Directions:

In the following we suppose, that we are at the k th iteration. Accordingly a point $\underline{x}_k \in L_p$ is given, and we know also the matrix $\underline{A}_1$ of the active constraints corresponding to $\underline{x}_k$. Then we determine the effective direction $\underline{r}_k$ belonging to this $\underline{x}_k$ in such a way, that we solve the problem,

$$(2.5.1) \quad \max \left\{ \underline{C}^T \underline{r} \mid \underline{r} \in R \right\}.$$

where the elements of R are the feasible directions belonging to $\underline{x}_k$. Hence the elements of R are also the ~~non-trivial~~ solutions of the inequality system

$$\underline{A}_1 \underline{r} \leq 0$$

These solutions are normalized in some way. Therefore, as we have already mentioned, the problem (2.5.1) surely has an optimal solution, which we denote by $\underline{r}_k$. It is also known, that in case of $\underline{C}^T \underline{r}_k > 0$ $\underline{r}_k$ is really an effective direction, but if $\underline{C}^T \underline{r}_k \leq 0$, then there does not belong any effective direction to $\underline{x}_k$.

Then according to the way we normalize the elements of R , we come to different variants of the problem (2.5.1). In the following these variants will be discussed.

I. First we apply the relation $|\underline{r}|=1$. Then the equivalent of (2.5.1) is the problem

$$(2.5.2) \quad \begin{aligned} \underline{A}_1 \underline{r} &\leq \underline{0} \\ |\underline{r}| &= 1 \\ \underline{C}^T \underline{r} &\longrightarrow \max \end{aligned}$$

This is a particular non-linear programming problem, which can be reduced to a quadratic programming problem of known type. And if there are no active restrictions, i.e. $\underline{x}_k$ is an inner point of L_p , the problem (2.5.2) reduces to the problem

$$\max \left\{ \underline{C}^T \underline{r} \mid |\underline{r}| = 1 \right\}$$

whose trivial solution is the vector

$$\underline{r}_k = \frac{1}{|\underline{C}|} \cdot \underline{C}$$

II. Computationally it is much more advantageous to apply the restriction

$$-1 \leq \underline{r} \leq 1$$

for normalization. The problem to be solved is then the linear programming problem.

$$(2.5.3) \quad \begin{aligned} \underline{A}_1 \underline{r} &\leq \underline{0} \\ -1 \leq \underline{r} \leq 1 \\ \underline{C}^T \underline{r} &\longrightarrow \max \end{aligned}$$

which can be transformed by means of $\underline{r} = \underline{y} - 1$ into the problem

$$(2.5.4) \quad \begin{aligned} 0 \leq \underline{y} \leq 2 \cdot 1 \\ \underline{A}_1 \underline{y} \leq \underline{A}_1 \cdot 1 \\ \underline{C}^T \underline{y} \longrightarrow \max \end{aligned}$$

If $\underline{y}_k$ is its optimal solution, then $\underline{r}_k = \underline{y}_k^{-1}$ represents an optimal solution for the problem (2.5.3). Naturally $\underline{r}_k$ is an effective direction only in case of $\underline{C}^T \underline{r}_k > 0$.

If it happens that there is no active constraint, then the problem (2.5.4) can be solved also in a trivial way. Namely, it can be easily seen, that then the vector $\underline{y}_k$, whose i th component is given by

$$\underline{e}_i^T \underline{y}_k = \begin{cases} 2 & , \text{ if } \underline{e}_i^T \underline{C} > 0 \\ 0 & , \text{ if } \underline{e}_i^T \underline{C} \leq 0 \end{cases}$$

is an optimal solution. The corresponding effective direction $\underline{r}_k$ can be given by the formula $\underline{r}_k = \underline{y}_k^{-1}$.

III. This variant can be characterized in such a way, that the elements of R are normalized by the restriction $|\underline{Q} \underline{r}| = 1$, where $\underline{Q}$ is a non-singular matrix. We impose, that $\underline{Q}$ should take the form

$$(2.5.5) \quad \underline{Q} = \begin{bmatrix} \underline{A} & \underline{1} \\ \underline{B} \end{bmatrix}$$

In connection with $\underline{A}_1$ we must require, that the row vectors of $\underline{A}_1$ form a linearly independent system. As relating to $\underline{B}$, for the time being we require only that $\underline{Q}$ be invertible. (Later we shall solve our problem with specially chosen matrices $\underline{B}$).

Then we have to solve the following problem:

$$(2.5.6) \quad \begin{aligned} & \underline{A}_1 \underline{r} \leq 0 \\ & |\underline{Q} \underline{r}| = 1 \\ & \underline{C}^T \underline{x} \longrightarrow \max \end{aligned}$$

For the sake of this we define the vector

$$(2.5.7) \quad \underline{c}^T \underline{Q}^{-1} = [\underline{y}^T, \underline{z}^T]$$

Where the number of the components of $\underline{y}$ is equal to the number of the active constraints, and $\underline{z}$ has as many components as the number of the row vectors of $\underline{B}$.

Then from $\underline{y}$ we construct the vector $\underline{y}^*$, whose i th component is given by

$$(2.5.8) \quad \underline{e}_i^T \underline{y}^* = \begin{cases} \underline{e}_i^T \underline{y}, & \text{if } \underline{e}_i^T \underline{y} < 0 \\ 0, & \text{if } \underline{e}_i^T \underline{y} \geq 0 \end{cases}$$

Then we define the vector $\underline{s}$, which plays an important part in our reasoning:

$$(2.5.9) \quad \underline{s} = \begin{bmatrix} \underline{y}^* \\ \underline{z} \end{bmatrix}$$

This vector $\underline{s}$ has remarkable properties.

Theorem 2.8:

If $\underline{s} = \underline{0}$, then $\underline{x}_k$ is an optimal point.

Theorem 2.9:

If $\underline{s} \neq \underline{0}$, then the vector $\underline{r}_k = \frac{1}{|\underline{s}|} \underline{Q}^{-1} \underline{s}$ is an effective direction, and at the same time is an optimal solution of the problem (2.5.6).

Now we can give the following procedure for the determination of the effective direction $\underline{r}_k$:

1. On the basis of (2.5.7) we determine the vector $\begin{bmatrix} \underline{y}^T, \underline{z}^T \end{bmatrix}$.
2. According to (2.5.8) we construct the vector $\underline{y}$ from $\underline{y}$, then we write the vector $\underline{s}^T = \begin{bmatrix} \underline{y}^T, \underline{z}^T \end{bmatrix}$.
3. If $\underline{s} = \underline{0}$, then $\underline{x}_k$ is an optimal solution.
4. If $\underline{s} \neq \underline{0}$, then $\underline{r}_k = \frac{1}{|\underline{s}|} \underline{Q}^{-1} \underline{s}$ is the wanted effective direction.

A great advantage of the presented procedure is, that it shows a direct possibility for the determination of the effective directions. But it has the disadvantage, that in case of large scale problems the inversion of $\underline{Q}$ may give rise to severe difficulties, since the order of $\underline{Q}$ is equal to the number of the unknowns. But these difficulties can be decreased considerably by suitably choosing the matrix $\underline{B}$.

IV. An it is known, the matrix $\underline{A}_1$ can be transformed by the usual simplex transformation into the form

$$(2.5.10) \quad \underline{A}_1 = \underline{C} \begin{bmatrix} \underline{E}_r & \underline{D} \end{bmatrix} = \begin{bmatrix} \underline{C} & \underline{CD} \end{bmatrix}$$

Where $\underline{C}$ is a non-singular block of order r . On the basis of this the basis of this the choice

$$(2.5.11) \quad \underline{B} = \begin{bmatrix} \underline{0} & \underline{E}_d \end{bmatrix}$$

Where d is the number of the row vectors of $\underline{B}$. Then

$$\underline{Q} = \begin{bmatrix} \underline{C} & \underline{CD} \\ \underline{0} & \underline{E}_d \end{bmatrix}$$

and as it is easy to verify,

$$\underline{Q}^{-1} = \begin{bmatrix} \underline{C}^{-1} & -\underline{D} \\ \underline{0} & \underline{E}_d \end{bmatrix}$$

So, with the partitioning $\underline{c}^T = [\underline{c}_1^T, \underline{c}_2^T]$

$$\underline{c}^T \underline{Q}^{-1} = [\underline{c}_1^T \underline{C}^{-1}, \underline{c}_2^T - \underline{c}_1^T \underline{D}]$$

Therefore,

$$\underline{y}^T = \underline{c}_1^T \underline{C}^{-1}$$

and

$$\underline{z}^T = \underline{c}_2^T - \underline{c}_1^T \underline{D}.$$

On this basis it is easy to construct the vectors $\underline{y}^T$ and $\underline{s}^T = [\underline{y}^T, \underline{z}^T]$. If $\underline{s} = \underline{0}$, then $\underline{x}_k$ is an optimal point, otherwise

$$(2.5.12) \quad \underline{r}_k = \frac{1}{|\underline{s}|} \hat{\underline{r}}_k$$

where

$$(2.5.13) \quad \hat{\underline{r}}_k = \begin{bmatrix} \underline{C}^{-1} \\ \underline{0} \end{bmatrix} \underline{y} + \begin{bmatrix} -\underline{D} \\ \underline{E}_d \end{bmatrix} \underline{z}.$$

is the wanted effective direction. This $\hat{\underline{r}}_k$ can be written also in the form

$$\hat{\underline{r}}_k = \underline{n}_k + \underline{t}_k$$

where

$$\underline{n}_k = \begin{bmatrix} \underline{C}^{-1} \\ \underline{0} \end{bmatrix} \underline{y} \quad \text{and} \quad \underline{t}_k = \begin{bmatrix} -\underline{D} \\ \underline{E}_d \end{bmatrix} \underline{z}$$

This resolution of $\hat{\underline{r}}_k$ is justified by the fact, that the vector $\underline{t}_k$, whose determination is relatively simple, has an important property :

Theorem 2.10:

If $\underline{z} \neq \underline{0}$, then $\underline{t}_k$ is an effective direction.

On this basis the following procedure can be constructed:

1. We perform the factorization $\underline{A}_1 = \underline{C} \begin{bmatrix} \underline{E}_r & \underline{D} \end{bmatrix}$ of the matrix $\underline{A}_1$.
2. We determine the vector $\underline{z} = \underline{c}_2 - \underline{D}^T \underline{c}_1$.
3. If $\underline{z} \neq \underline{0}$, then $\underline{t}_k = \begin{bmatrix} -\underline{D} \\ \underline{E}_d \end{bmatrix} \underline{z}$ is an effective direction.
4. If $\underline{z} = \underline{0}$, then we determine the vector $\underline{y}^T = \underline{c}_1^T \underline{C}^{-1}$ and from it we construct the vector $\underline{y}$.
5. If $\underline{y} = \underline{0}$, then $\underline{x}_k$ is an optimal solution.
6. If $\underline{y} \neq \underline{0}$, then $\underline{n}_k = \begin{bmatrix} \underline{C}^{-1} \\ \underline{0} \end{bmatrix} \underline{y}$ is an effective direction.

V. We get a more complicated case than the previous one, if for

$$\underline{A}_1 = \begin{bmatrix} \underline{C} & \underline{CD} \end{bmatrix}$$

we give the matrix $\underline{B}$ in the form

$$\underline{B} = \begin{bmatrix} -\underline{D}^T & \underline{E}_d \end{bmatrix}$$

in this case the relation

$$(2.5.14) \quad \underline{A}_1 \underline{B}^T = \underline{0}$$

holds, which means, that the row vector space of $\underline{B}$ is the orthogonal complement of the row vector space of $\underline{A}_1$. Under this choice

$$\underline{Q} = \begin{bmatrix} \underline{C} & \underline{CD} \\ -\underline{D}^T & \underline{E}_d \end{bmatrix}$$

and

$$\underline{Q}^{-1} = \begin{bmatrix} (\underline{E}_r - \underline{DS}^{-1} \underline{D}^T) \underline{C}^{-1} & -\underline{DS}^{-1} \\ \underline{S}^{-1} \underline{D}^T \underline{C}^{-1} & \underline{S}^{-1} \end{bmatrix}$$

where $\underline{S} = \underline{E}_d + \underline{D}^T \underline{D} = \underline{S}^T$

So, according to (2.5.7) - depending on the partition $\underline{C}^T = [\underline{c}_1^T, \underline{c}_2^T]$ -

$$\underline{y}^T = [(\underline{c}_2^T - \underline{c}_1^T \underline{D}) \underline{S}^{-1} \underline{D}^T + \underline{c}_1^T] \underline{C}^{-1}$$

and

$$(2.5.15) \quad \underline{z}^T = (\underline{c}_2^T - \underline{c}_1^T \underline{D}) \underline{S}^{-1}$$

By using this last relation, $\underline{y}$ can be written in the form

$$(2.5.16) \quad \underline{y}^T = (\underline{z}^T \underline{D}^T + \underline{c}_1^T) \underline{C}^{-1}$$

Similarly as before, we can apply the resolution

$$(2.5.17) \quad \hat{\underline{r}}_k = \underline{n}_k + \underline{t}_k$$

where in this case

$$\underline{n}_k = \begin{bmatrix} \underline{E}_r - \underline{D} \underline{S}^{-1} \underline{D}^T \\ \underline{S}^{-1} \underline{D}^T \end{bmatrix} \cdot \underline{C}^{-1} \underline{y}$$

and,

$$\underline{t}_k = \begin{bmatrix} -\underline{D} \\ \underline{E}_d \end{bmatrix} \underline{S}^{-1} \underline{z}$$

where we form the vector $\underline{y}$ from $\underline{y}$. Also here $\underline{t}_k$ has a similar property, that

Theorem 2.11:

If $\underline{z} \neq 0$, then $\underline{t}_k$ is an effective direction.

Then we can formulate a similar procedure, as in case IV:

1. We perform the factorization $\underline{A}_1 = \underline{C} [\underline{E}_r, \underline{D}]$
2. We determine the vector $\underline{z}^T = (\underline{c}_2^T - \underline{c}_1^T \underline{D}) \underline{S}^{-1}$
3. If $\underline{z} \neq 0$, then $\underline{t}_k = \begin{bmatrix} -\underline{D} \\ \underline{E}_d \end{bmatrix} \underline{S}^{-1} \underline{z}$ as an effective direction.

4. If $\underline{z} = \underline{0}$, then we determine the vector $\underline{y}$ on the basis of the relation

$$\underline{y}^T = \underline{c}_1^T \underline{C}^{-1}, \text{ and then the vector } \underline{y}.$$

5. If $\underline{y} = \underline{0}$, then $\underline{x}_k$ is an optimal solution.

6. If $\underline{y} \neq \underline{0}$, then the vector $\underline{n}_k = \begin{bmatrix} \underline{E}_r - \underline{D} \underline{S}^{-1} \underline{D}^T \underline{C}^{-1} \underline{y} \\ \underline{S}^{-1} \underline{D}^T \end{bmatrix}$

is an effective direction.

Chapter 3

The Indirect Methods

3.1. The Principle of Decomposition:

We shall examine the following problem:

$$\begin{aligned}
 & \underline{x}_1 \geq \underline{0} \quad , \quad \underline{x}_2 \geq \underline{0} \quad , \quad \dots \quad , \quad \underline{x}_p \geq \underline{0} \\
 & \underline{K}_1 \underline{x}_1 + \underline{K}_2 \underline{x}_2 + \dots + \underline{K}_p \underline{x}_p \leq \underline{b}_0 \\
 (3.1.1) \quad & \underline{A}_1 \underline{x}_1 \leq \underline{b}_1 \\
 & \underline{A}_2 \underline{x}_2 \leq \underline{b}_2 \\
 & \quad \quad \quad \cdot \quad \cdot \quad \cdot \\
 & \quad \quad \quad \underline{A}_p \underline{x}_p \leq \underline{b}_p
 \end{aligned}$$

$$\underline{c}_1^T \underline{x}_1 + \underline{c}_2^T \underline{x}_2 + \dots + \underline{c}_p^T \underline{x}_p \longrightarrow \max$$

It can be considered as a special case of the problem (2.3.1) with $(\underline{K}_0 = \underline{0})$

If the number of the components of the vectors $\underline{b}_0, \underline{b}_1, \underline{b}_2, \dots, \underline{b}_p$ are $m_0, m_1, m_2, \dots, m_p$, then the number of the central constraints is m_0 , while the number of the sector constraints is $m_1 + m_2 + \dots + m_p = m$.

At the same time let us assume, that the vectors $\underline{x}_1, \underline{x}_2, \dots, \underline{x}_p$ showing the level of the sector activities contain $n_1, n_2, \dots, n_p$ components. The matrix $\underline{K}_i$ is of type $m_0 \cdot n_i$, and the matrix $\underline{A}_i$ is of type $m_i \cdot n_i$, where $i=1, 2, \dots, p$. Then the problem (3.1.1) is such a linear programming problem, where the number of the unknowns is $n_1 + n_2 + \dots + n_p$, and the number of the constraints is $m_0 + m_1 + m_2 + \dots + m_p$. We suppose, that the set

$$(3.1.2) \quad L_i = \{ \underline{x}_i \mid \underline{x}_i \geq \underline{0}, \underline{A}_i \underline{x}_i \leq \underline{b}_i \} \quad (i=1, \dots, p)$$

of the so-called sector programs is non-empty and bounded. The principle of the decomposition introduced by DANTZIG and WOLFE, is based on the fact that under the given conditions the set of the sector programs can be written also in the form (3.1.3) $L_i = \{ \underline{x}_i \mid \underline{x}_i = \underline{P}_i \underline{V}_i, \underline{V}_i \geq \underline{0}, \underline{1}^T \underline{V}_i = 1 \}$ ($i=1, 2, \dots, p$)

Where $\underline{P}_i$ is the matrix constructed from the vectors corresponding to the extreme points of L_i , and $\underline{V}_i$ is the frequency vector. If we apply the relation (3.1.3) to the problem (3.1.1), then we obtain the following transformed version of this problem:

$$\begin{aligned} & \underline{V}_1 \geq \underline{0}, \underline{V}_2 \geq \underline{0}, \dots, \underline{V}_p \geq \underline{0} \\ & (\underline{K}_1 \underline{P}_1) \underline{V}_1 + (\underline{K}_2 \underline{P}_2) \underline{V}_2 + \dots + (\underline{K}_p \underline{P}_p) \underline{V}_p \leq \underline{b}_0 \\ & \underline{1}^T \underline{V}_1 = 1 \\ & \underline{1}^T \underline{V}_2 = 1 \\ & \underline{1}^T \underline{V}_p = 1 \\ (3.1.4) \quad & (\underline{C}_1^T \underline{P}_1) \underline{V}_1 + (\underline{C}_2^T \underline{P}_2) \underline{V}_2 + \dots + (\underline{C}_p^T \underline{P}_p) \underline{V}_p \longrightarrow \max \end{aligned}$$

It can be easily seen, that the two problems are equivalent to each other in the following sense:

Theorem 3.1:

If the vector system $(\underline{V}_1^0, \underline{V}_2^0, \dots, \underline{V}_p^0)$ represents an optimal solution for the problem (3.1.4.), then the vector system $(\underline{x}_1^0, \underline{x}_2^0, \dots, \underline{x}_p^0)$ is an

optimal solution of the problem (3.1.1) where

$$\underline{x}_i^0 = \underline{P}_i \underline{V}_i^0 \quad (i=1,2,\dots,P)$$

Now we formulate the dual pair of (3.1.4)

$$\begin{aligned} \underline{S} \geq 0, \quad y_1 \in E_1, \quad y_2 \in E_1, \dots, y_p \in E_1 \\ (3.1.5) \quad \begin{aligned} \underline{S}^T \underline{K}_1 \underline{P}_1 + y_1 \underline{1}^T &\geq \underline{C}_1^T \underline{P}_1 \\ \underline{S}^T \underline{K}_2 \underline{P}_2 + y_2 \underline{1}^T &\geq \underline{C}_2^T \underline{P}_2 \\ \underline{S}^T \underline{K}_p \underline{P}_p + y_p \underline{1}^T &\geq \underline{C}_p^T \underline{P}_p \\ \underline{S}^T \underline{b}_0 + y_1 + y_2 + \dots + y_p &\longrightarrow \min \end{aligned} \end{aligned}$$

The components of $\underline{S}$ can be interpreted as the shadow prices of the central resources, while the scalars $y_1, y_2, \dots, y_p$ represent the efficiency bound of the sectors.

Now, in the t th iteration step, such a reduced version:

$$\begin{aligned} \hat{\underline{V}}_1 \geq 0, \quad \hat{\underline{V}}_2 \geq 0, \dots, \hat{\underline{V}}_p \geq 0 \\ (3.1.6) \quad \begin{aligned} (\underline{K}_1 \underline{P}_1)^T \hat{\underline{V}}_1 + (\underline{K}_2 \underline{P}_2)^T \hat{\underline{V}}_2 + \dots + (\underline{K}_p \underline{P}_p)^T \hat{\underline{V}}_p &\leq \underline{b}_0 \\ \underline{1}^T \hat{\underline{V}}_1 &= 1 \\ \underline{1}^T \hat{\underline{V}}_2 &= 1 \\ &\vdots \\ \underline{1}^T \hat{\underline{V}}_p &= 1 \\ (\underline{C}_1^T \underline{P}_1)^T \hat{\underline{V}}_1 + (\underline{C}_2^T \underline{P}_2)^T \hat{\underline{V}}_2 + \dots + (\underline{C}_p^T \underline{P}_p)^T \hat{\underline{V}}_p &\longrightarrow \max \end{aligned} \end{aligned}$$

of the problem (3.1.4) is given. Here the column vectors of $\hat{\underline{P}}_i$ form a subset of the column vectors of $\underline{P}_i$ and consequently the components of $\hat{\underline{V}}_i$ are similarly a subset of the components of $\underline{V}_i$.

Let the optimal solution of (3.1.6) be the vector system $(\hat{\underline{V}}_{1t}, \hat{\underline{V}}_{2t}, \dots, \hat{\underline{V}}_{pt})$, and the corresponding optimal solution of the dual of (3.1.6) be the vector system $(\underline{S}_t, y_{1t}, y_{2t}, \dots, y_{pt})$. According to the strong duality theorem

$$\begin{aligned} (3.1.7) \quad (\underline{C}_1^T \hat{\underline{P}}_1) \hat{\underline{V}}_{1t} + (\underline{C}_2^T \hat{\underline{P}}_2) \hat{\underline{V}}_{2t} + \dots + (\underline{C}_{p-p}^T \hat{\underline{P}}_{p-p}) \hat{\underline{V}}_{pt} = \\ = \underline{S}_t^T \underline{b}_0 + y_{1t} + y_{2t} + \dots + y_{pt}. \end{aligned}$$

An important relation with respect to the two solution systems is expressed by

Theorem 3.2:

If $(\underline{S}_t, y_{1t}, \dots, y_{pt})$ is a feasible solution of the problem (3.1.5), then the vector system $(\underline{x}_{1t}, \underline{x}_{2t}, \dots, \underline{x}_{pt})$, where

$$\underline{x}_{it} = \hat{\underline{P}}_i \hat{\underline{V}}_{it} \quad (i=1, 2, \dots, p)$$

represents an optimal solution for the problem (3.1.1).

This criterium cannot be utilized directly, since the problem (3.1.5) is not available, we know only a reduced version of it. But the indirect application of the criterium is possible. Therefore, let us construct the so-

called sector problem:

$$(3.1.8) \quad \begin{aligned} \underline{x}_i &\geq \underline{0} \\ \underline{A}_i \underline{x}_i &\leq \underline{b}_i \quad (i=1, 2, \dots, p) \\ \underline{Z}_i &= (\underline{C}_i^T - \underline{S}_t^T \underline{K}_i) \underline{x}_i \longrightarrow \max \end{aligned}$$

Let $\hat{\underline{x}}_{it}$ be an optimal basic solution of the problem (3.1.8), and let

$$\underline{Z}_{it} = (\underline{C}_i^T - \underline{S}_t^T \underline{K}_i) \hat{\underline{x}}_{it}$$

Knowing the values $\underline{Z}_{it}$ there is already a simple way for the examination of the criterium.

Theorem 3.3.:

If $\underline{Z}_{it} \leq \underline{y}_{it}$ holds true for every possible i , then the criterium formulated in theorem 3.2 is satisfied.

If it happens that the criterium $\underline{Z}_{it} \leq \underline{y}_{it}$ does not hold for all i , then the reduced problem must be extended further, that the matrix $[\hat{\underline{P}}_{it}, \hat{\underline{x}}_{it}]$ represents a suitable extension of the matrix $\hat{\underline{P}}_{it}$. After the execution of this extension we can obtain such a new reduced problem.

Now we can formulate the algorithm:

0. On the basis of the foregoing we construct an initial reduced problem. This is the problem (3.1.6) for the step $t = 0$

1. We solve the reduced problem corresponding to (3.1.6)

Let the vector system $(\hat{V}_{1t}, \dots, \hat{V}_{pt}; \hat{S}_t, y_{1t}, \dots, y_{pt})$ be a solution.

2. We solve one after the other the sector problems corresponding to (3.1.8)

We denote the solution by the symbol $(\hat{x}_{it}, z_{it})$, where $\hat{x}_{it}$ is in any case a basic solution. If for some i , $z_{it} > y_{it}$, then we replace the matrix $\hat{P}_i$ by the matrix $[\hat{P}_i, \hat{x}_{it}]$, and writing $t+1$ in place of t return to step 1.*

3. If $z_{it} \leq y_{it}$ for all i , then the vector system $(x_{1t}, \dots, x_{pt})$, where

$$x_{it} = \hat{P}_i \hat{V}_{it}, \quad (i=1,2,\dots,p)$$

represents an optimal solution for the original problem. STOP.

3.2. The Relaxation Procedure:

The main point of the relaxation is that in the calculations we permit a part of the constraints to be violated. We judge by the solution of the reduced problem obtained in this way whether we have already get the optimal

* Of course we could proceed also in such a way, that we would solve every sector problem, and extend the matrix $\hat{P}_i$ in every case when $z_{it} > y_{it}$.

solution. If the optimality does not come true, then we repeat the calculations with a new set of the relaxed constraints and so on. By a suitable organization of the relaxation we can reach to the optimal solution of the whole problem in a finite number of steps, if it exists.

In the following we shall present a relaxation procedure, such that applies the relaxation exclusively for the non-negativity constraints. This procedure is due to ROSEN.

Let us consider the problem given by (3.1.1). If we take into consideration the complete form of the sector problems

$$(3.2.1.) \quad \underline{B}_i \underline{y}_i = \underline{b}_i \quad (i=1,2,\dots,p)$$

where

$$\underline{B}_i = \begin{bmatrix} \underline{A}_i & \underline{E}_{m_i} \end{bmatrix} \text{ and } \underline{y}_i^T = \begin{bmatrix} \underline{x}_i^T & \underline{u}_i^T \end{bmatrix},$$

and in addition, we apply also the notations

$$\underline{K}_i' = \begin{bmatrix} \underline{K}_i & \underline{o} \end{bmatrix} \quad \text{and} \quad \underline{C}_i'^T = \begin{bmatrix} \underline{c}_i^T & \underline{o}^T \end{bmatrix}$$

then we can write our problem in the following way:

$$\underline{y}_1 \geq \underline{o}, \quad \underline{y}_2 \geq \underline{o}, \dots, \underline{y}_p \geq \underline{o}$$

$$\underline{K}_1' \underline{y}_1 + \underline{K}_2' \underline{y}_2 + \dots + \underline{K}_p' \underline{y}_p \leq \underline{b}_0$$

(3.2.2)

$$\underline{B}_1 \underline{y}_1 = \underline{b}_1$$

$$\underline{B}_2 \underline{y}_2 = \underline{b}_2$$

$$\underline{B}_p \underline{y}_p = \underline{b}_p$$

$$\underline{C}_1'^T \underline{y}_1 + \underline{C}_2'^T \underline{y}_2 + \dots + \underline{C}_p'^T \underline{y}_p \longrightarrow \max$$

From the construction corresponding to (3.2.1) it follows, that the row vectors for each of the matrices $\underline{B}_i$ form a linearly independent system. We suppose that the set of the feasible sector programs is non-empty and bounded.

Now let us assume, that by a possible rearrangement of the columns every matrix $\underline{B}_i$ can be partitioned in the way

$$[\underline{B}_{i1}, \underline{B}_{i2}] \quad (i=1,2,\dots,p)$$

where the block $\underline{B}_{i1}$ is non-singular, so that also the relation

$$(3.2.3) \quad \hat{\underline{b}}_i = \underline{B}_{i1}^{-1} \underline{b}_i \geq 0$$

be valid. This means, that $\begin{bmatrix} \hat{\underline{b}}_i \\ 0 \end{bmatrix}$ represents such a feasible basic solution on the set of the sector programs, where does not occur the degeneracy. Then in accordance with the partition of $\underline{B}_i$ we apply the following further partitions:

$$\begin{aligned} \underline{y}_i^T &= [\underline{y}_{i1}^T, \underline{y}_{i2}^T] \\ \underline{K}_i &= [\underline{K}_{i1}, \underline{K}_{i2}] \\ \underline{C}_i^T &= [\underline{C}_{i1}^T, \underline{C}_{i2}^T] \end{aligned}$$

On this basis we can also formulate our problem in the following form:

$$\begin{aligned} (3.2.4) \quad & \underline{y}_{11} \geq 0, \underline{y}_{12} \geq 0, \dots, \underline{y}_{p1} \geq 0, \underline{y}_{p2} \geq 0 \\ & \underline{K}_{11} \underline{y}_{11} + \underline{K}_{12} \underline{y}_{12} + \dots + \underline{K}_{p1} \underline{y}_{p1} + \underline{K}_{p2} \underline{y}_{p2} \leq \underline{b}_0 \\ & \underline{B}_{11} \underline{y}_{11} + \underline{B}_{12} \underline{y}_{12} \dots \underline{B}_{p1} \underline{y}_{p1} + \underline{B}_{p2} \underline{y}_{p2} = \underline{b}_1 \\ & \underline{C}_{11}^T \underline{y}_{11} + \underline{C}_{12}^T \underline{y}_{12} + \dots + \underline{C}_{p1}^T \underline{y}_{p1} + \underline{C}_{p2}^T \underline{y}_{p2} \rightarrow \max \end{aligned}$$

But from the assumption concerning $\underline{B}_i$ follows, that

$$(3.2.5) \quad \underline{y}_{i1} = \hat{\underline{b}}_i - \hat{\underline{B}}_{i2} \underline{y}_{i2} \quad (i=1,2,\dots,p),$$

where

$$\hat{\underline{B}}_{i2} = \underline{B}_{i1}^{-1} \underline{B}_{i2}$$

So by using these relations (3.2.4) can be rewritten as:

$$(3.2.6) \quad \begin{aligned} & \underline{y}_{i2} \geq 0, \dots, \underline{y}_{p2} \geq 0 \\ & \hat{\underline{K}}_{12} \underline{y}_{12} + \dots + \hat{\underline{K}}_{p2} \underline{y}_{p2} \leq \hat{\underline{b}}_0 \\ & \hat{\underline{C}}_{12}^T \underline{y}_{12} + \dots + \hat{\underline{C}}_{p2}^T \underline{y}_{p2} + \alpha \longrightarrow \max \end{aligned}$$

we call this problem the reduced problem belonging to the method, where.

$$(3.2.7) \quad \begin{aligned} \hat{\underline{K}}_{i2} &= \underline{K}_{12} - \underline{K}_{i1} \hat{\underline{B}}_{i2} \\ \hat{\underline{b}}_0 &= \underline{b}_0 - \underline{K}_{11} \hat{\underline{b}}_1 - \dots - \underline{K}_{p1} \hat{\underline{b}}_p \\ \hat{\underline{C}}_{i2}^T &= \underline{C}_{i2}^T - \underline{C}_{i1}^T \hat{\underline{B}}_{i2} \\ \alpha &= \underline{C}_{11}^T \hat{\underline{b}}_1 + \dots + \underline{C}_{p1}^T \hat{\underline{b}}_p \end{aligned}$$

Now we assume also, that the reduced problem (3.2.6) has an optimal basic solution. Let the vector system $(\underline{y}_{12}^0, \underline{y}_{22}^0, \dots, \underline{y}_{p2}^0)$ be such an optimal basic solution. On this basis, taking into consideration the relation (3.2.5), we can construct the vectors of the form

$$(3.2.8) \quad \underline{y}_{i1}^0 = \hat{\underline{b}}_i - \hat{\underline{B}}_{i2} \underline{y}_{i2}^0 \quad (i=1,2,\dots,p)$$

It is clear, that the vector system

$$(y_{11}^0, y_{12}^0, \dots, y_{p1}^0, y_{p2}^0)$$

is an optimal solution of the problem

$$y_{12} \geq 0, \dots, y_{p2} \geq 0$$

$$\frac{K_{-11}Y_{-11}}{-11} + \frac{K_{-12}Y_{-12}}{-12} + \dots + \frac{K_{-p1}Y_{-p1}}{-p1} + \frac{K_{-p2}Y_{-p2}}{-p2} \leq \frac{b_0}{-0}$$

(3.2.9)

$$B_{11}Y_{11} + B_{12}Y_{12} \leq b_1$$

$$\frac{B_{-p1}}{-p1} \frac{Y_{-p1}}{-p1} + \frac{B_{-p2}}{-p2} \frac{Y_{-p2}}{-p2} \leq \frac{b_p}{-p}$$

$$C_{-11}^T \frac{y}{-11} + C_{-12}^T \frac{y}{-12} + \dots + C_{-p1}^T \frac{y}{-p1} + C_{-p2}^T \frac{y}{-p2} \longrightarrow \max$$

which is the relaxed form of (3.2.4)

Theorem 3.4:

If the vector $\underline{y}_{i1}^0$ defined by (3.2.8) is non-negative for all i , then the vector system $(\underline{y}_{i1}^0, \underline{y}_{i2}^0, \dots, \underline{y}_{p1}^0, \underline{y}_{p2}^0)$ represents an optimal solution for the original problem.

If the condition $\underline{y}_{i1}^0 \geq 0$ does not hold, then we modify the reduced problem (3.2.9) in such a way, that we replace one of the components of $\underline{y}_{i2}$ for which the corresponding component of $\underline{y}_{i2}^0$ is positive by a component of $\underline{y}_{i1}$ for which the corresponding component of $\underline{y}_{i1}^0$ is negative.

Chapter 4

The Parametric Programming

We deal only with the case, when the given constants are linear functions of a single parameter t . We turn our attention first of all to the case, when only the components of the vectors $\underline{b}$ and $\underline{c}$, but not those of the coefficient matrix $\underline{A}$ are depending on t . The more general case, when even the elements of $\underline{A}$ depend on t is discussed later.

4.1. The Reduced Problem:

Let us consider the following problem:

$$(4.1.1) \quad \begin{aligned} \underline{x} &\geq \underline{0} \\ \underline{A} \underline{x} &\leq \underline{b} + t \underline{p} & (\underline{p} \neq \underline{0}) \\ (\underline{c}^T + t \underline{q}^T) \underline{x} &\rightarrow \max & (\underline{q} \neq \underline{0}) \end{aligned}$$

We suppose that in case of $t=0$ this problem has an optimal solution.

The following complete problem

$$(4.1.2) \quad \begin{aligned} \underline{x} &\geq \underline{0} \quad \underline{u} \geq \underline{0} \\ \underline{A} \underline{x} + \underline{u} &= \underline{b} + t \underline{p} \\ (\underline{c}^T + t \underline{q}^T) \underline{x} + \underline{o}^T \underline{u} &\rightarrow \max \end{aligned}$$

is equivalent to (4.1.1)

By introducing the notations

$$\underline{L} = \begin{bmatrix} \underline{A} & \underline{E} & \underline{b} \\ \underline{c}^T & \underline{o}^T & \underline{o} \end{bmatrix} \quad \text{and} \quad \underline{H} = \begin{bmatrix} \underline{0} & \underline{0} & \underline{p} \\ \underline{q}^T & \underline{b}^T & \underline{0} \end{bmatrix}$$

we associate the coefficient matrix

$$(4.1.3) \quad \underline{L}(t) = \underline{L} + t \underline{H}$$

with (4.1.2).

According to the condition $t=0$ the problem has an optimal solution, that is for the coefficient matrix $\underline{L}(0) = \underline{L}$.

Let us suppose, that the basis matrix

$$(4.1.4) \quad \underline{B} = \begin{bmatrix} \underline{D} & \underline{o} \\ \underline{d}^T & 1 \end{bmatrix}$$

is belonging to this optimal solution, where the column vectors of the block $\begin{bmatrix} \underline{D} \\ \underline{d}^T \end{bmatrix}$ are chosen from among the column vectors of

$$\text{the block } \begin{bmatrix} \underline{A} & \underline{E} \\ \underline{C}^T & \underline{o}^T \end{bmatrix}$$

Now we shall examine, for which values $t \geq 0$ remains the given basis optimal.

Therefore we construct the matrix

$$(4.1.5) \quad \underline{B}(t) = \underline{B} + t \hat{\underline{B}}$$

where

$$\hat{\underline{B}} = \begin{bmatrix} \underline{o} & \underline{o} \\ \underline{r}^T & \underline{o} \end{bmatrix}$$

Here the column vectors of the block $\begin{bmatrix} \underline{o} \\ \underline{r}^T \end{bmatrix}$ are the corresponding column vectors of the block $\begin{bmatrix} \underline{o} & \underline{o} \\ \underline{q}^T & \underline{o}^T \end{bmatrix}$ of the matrix $\underline{H}$. so

$$(4.1.7) \quad \underline{B}(t) = \begin{bmatrix} \underline{D} & \underline{o} \\ (\underline{d}^T + t \underline{r}^T) & 1 \end{bmatrix}$$

It is easy to verify, that

$$\underline{B}(t) = \underline{B} + t \underline{B} \hat{\underline{B}} = \underline{B} (\underline{E} + t \hat{\underline{B}})$$

and

$$\underline{B}^{-1}(t) = (\underline{E} - t \hat{\underline{B}}) \underline{B}^{-1}$$

It is also clear, that

$$(4.1.8) \quad \underline{B}^{-1}(t) = \begin{bmatrix} \underline{D}^{-1} & \underline{0} \\ -(\underline{d}_q^T + t \underline{r}^T) \underline{D}^{-1} & \underline{1} \end{bmatrix}$$

The matrix $\underline{L}(t)$ in the basis corresponding to $\underline{B}(t)$ takes the form

$$(4.1.9) \quad \underline{B}^{-1}(t) \cdot \underline{L}(t) = (\underline{E} - t \hat{\underline{B}}) \underline{B}^{-1} (\underline{L} + t \underline{H})$$

This by means of (4.1.8), and on the basis of the basis of the definition of $\underline{L}$ and $\underline{H}$, can be written also in the form:

$$(4.1.10) \quad \underline{B}^{-1}(t) \cdot \underline{L}(t) = \underline{D}^{-1} \underline{A} \quad \underline{D}^{-1} \quad \underline{D}^{-1}(\underline{b} + t \underline{p}) \\ \underline{C}^T - \underline{d}^T \underline{D}^{-1} \underline{A} + t(\underline{q}^T - \underline{r}^T \underline{D}^{-1} \underline{A}) - (\underline{d}^T + t \underline{r}^T) \underline{D}^{-1} - (\underline{d}^T + t \underline{r}^T) \underline{D}^{-1}(\underline{b} + t \underline{p})$$

The corresponding tableau is optimal for every t , for which the following relations are valid.

$$(4.1.11) \quad \underline{y}_p(t) = \underline{D}^{-1} \underline{b} + t \underline{D}^{-1} \underline{p} \geq \underline{0} \\ \underline{y}_{d1}^T(t) = \underline{d}^T \underline{D}^{-1} \underline{A} - \underline{C}^T + t(\underline{r}^T \underline{D}^{-1} \underline{A} - \underline{q}^T) \geq \underline{0}^T \\ \underline{y}_{d2}^T(t) = \underline{d}^T \underline{D}^{-1} + t \underline{r}^T \underline{D}^{-1} \geq \underline{0}^T$$

To obtain these formulas, we construct the initial matrix

$$(4.1.12) \quad \hat{\underline{C}}_0 = \begin{bmatrix} \underline{L} & \underline{P} \\ \underline{q}^T & \underline{0} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{E} & \underline{b} & | & \underline{P} \\ \underline{C}^T & \underline{0}^T & \underline{0} & | & \underline{0} \\ \underline{q}^T & \underline{0}^T & \underline{0} & | & \underline{0} \end{bmatrix}$$

and the corresponding basis matrix

$$\tilde{B} = \begin{bmatrix} \underline{B} & \underline{0} \\ \underline{\hat{r}}^T & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \underline{D} & \underline{0} & \underline{0} \\ \underline{d}^T & 1 & \underline{0} \\ \hline \underline{\hat{r}}^T & \underline{0} & 1 \end{array} \right]$$

Since

$$\begin{bmatrix} \underline{B} & \underline{0} \\ \underline{\hat{r}}^T & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \underline{B}^{-1} & \underline{0} \\ -\underline{\hat{r}}^T \underline{B}^{-1} & 1 \end{bmatrix} = \left[\begin{array}{cc|c} \underline{D}^{-1} & \underline{0} & \underline{0} \\ -\underline{d}^T \underline{D}^{-1} & 1 & \underline{0} \\ \hline -\underline{\hat{r}}^T \underline{D}^{-1} & \underline{0} & 1 \end{array} \right]$$

therefore

$$(4.1.13) \quad \underline{C}_1 = \tilde{B}^{-1} \hat{\underline{C}}_0 = \begin{bmatrix} \underline{B}^{-1} \underline{L} & \underline{B}^{-1} \underline{p} \\ \underline{q}^T - \underline{\hat{r}}^T \underline{B}^{-1} \underline{L} & -\underline{\hat{r}}^T \underline{B}^{-1} \underline{p} \end{bmatrix} \\ = \left[\begin{array}{ccc|c} \underline{D}^{-1} \underline{A} & \underline{D}^{-1} & \underline{D}^{-1} \underline{b} & \underline{D}^{-1} \underline{p} \\ \underline{c}^T - \underline{d}^T \underline{D}^{-1} \underline{A} & -\underline{d}^T \underline{D}^{-1} & -\underline{d}^T \underline{D}^{-1} \underline{b} & -\underline{d}^T \underline{D}^{-1} \underline{p} \\ \hline \underline{q}^T - \underline{r}^T \underline{D}^{-1} \underline{A} & -\underline{r}^T \underline{D}^{-1} & -\underline{r}^T \underline{D}^{-1} \underline{b} & -\underline{r}^T \underline{D}^{-1} \underline{p} \end{array} \right]$$

Since in (4.1.11) $\underline{y}_p(t)$ is the solution of the primal problem, while $\underline{y}_d^T(t) = [\underline{y}_{d1}^T(t), \underline{y}_{d2}^T(t)]$ represents the solution of the dual problem, therefore for the examination of the optimality two constants must be determined

$$(4.1.14) \quad t_p = \sup \{ t \mid \underline{y}_p(t) \geq \underline{0} \}$$

$$(4.1.15) \quad t_d = \sup \{ t \mid \underline{y}_d(t) \geq \underline{0} \}$$

$$\text{Then, if } \hat{t} = \min \{ t_p, t_d \}$$

That $\hat{t}$ represents the maximal value of the parameter t , for which the formulas of (4.1.11) still yield an optimal solution.

4.2. The Extension of the Examination:

Now we shall consider the following problem:

$$(4.2.1.) \quad \begin{aligned} \underline{x} &\geq \underline{0} \\ (\underline{A} + t\underline{M}) \underline{x} &\leq \underline{b} + t\underline{p} \\ (\underline{C}^T + t\underline{q}^T) \underline{x} &\longrightarrow \max \end{aligned}$$

Applying the notations introduced in point (4.1), we can write, that

$$\underline{L} = \begin{bmatrix} \underline{A} & \underline{E} & \underline{b} \\ \underline{C}^T & \underline{0}^T & \underline{0} \end{bmatrix}$$

$$\underline{H} = \begin{bmatrix} \underline{M} & \underline{0} & \underline{P} \\ \underline{q}^T & \underline{0}^T & \underline{0} \end{bmatrix}$$

and

$$\underline{L}(t) = \underline{L} + t \underline{H}$$

We suppose that we know a factorization of $\underline{H}$, that

$$\underline{H} = \underline{G} \cdot \underline{R}$$

where the number of columns of $\underline{G}$, resp. the number of rows of $\underline{R}$ is equal to the rank of $\underline{H}$. We suppose also, that in case $t=0$, there exists an optimal solution for our problem, and the matrix $\underline{B}$ gives an optimal basis.

This means that the simplex tableau corresponding to the relation $\underline{B}^{-1} \underline{L}$ is optimal. Now to the matrix $\underline{L}(t)$ we construct the corresponding matrix

$$\underline{B}(t) = \underline{B} + t \hat{\underline{B}}$$

Where $\hat{\underline{B}}$ can be written also in the form

$$\hat{\underline{B}} = \underline{G} \cdot \hat{\underline{R}}$$

So we have

$$(4.2.2.) \quad \underline{B}(t) = \underline{B} + t \underline{G} \hat{\underline{R}}$$

The inverse of $\underline{B}(t)$ can be written in the form

$$(4.2.3.) \quad \underline{B}^{-1}(t) = \underline{B}^{-1} - t \underline{B}^{-1} \underline{G} \underline{K}^{-1} \hat{\underline{R}} \underline{B}^{-1}$$

where

$$(4.2.4.) \quad \underline{K} = \underline{E} + t \hat{\underline{R}} \underline{B}^{-1} \underline{G}$$

In the basis determined by $\underline{B}(t)$, $\underline{L}(t)$ takes the form $\underline{B}^{-1}(t) \cdot \underline{L}(t)$

then

$$(4.2.5.) \quad \underline{B}^{-1}(t) \cdot \underline{L}(t) = \underline{B}^{-1} \underline{L} + t \underline{B}^{-1} \underline{G} \underline{K}^{-1} (\hat{\underline{R}} - \hat{\underline{R}} \underline{B}^{-1} \underline{L})$$

From the numerical point of view the first step is the factorization of $\underline{H}$ according to the given relation $\underline{H} = \underline{G} \cdot \underline{R}$. Then we construct the initial

tableau

$$(4.2.6.) \quad \hat{\underline{C}}_0 = \begin{bmatrix} \underline{L} & \underline{G} \\ \underline{R} & \underline{O} \end{bmatrix}$$

for which the basis matrix

$$(4.2.7.) \quad \tilde{\underline{B}} = \begin{bmatrix} \underline{B} & \underline{O} \\ \hat{\underline{R}} & \underline{E} \end{bmatrix}$$

By means of these we get the matrix

$$(4.2.8) \quad \tilde{\underline{B}}^{-1} \hat{\underline{C}}_0 = \underline{C}_1 = \begin{bmatrix} \underline{B}^{-1} \underline{L} & \underline{B}^{-1} \underline{G} \\ \underline{R} - \hat{\underline{R}} \underline{B}^{-1} \underline{L} & -\hat{\underline{R}} \underline{B}^{-1} \underline{G} \end{bmatrix}$$

Which already contains the necessary data for the numerical determination of the matrix $\underline{B}^{-1}(t)$. $\underline{L}(t)$, we need the determination of the inverse of $\underline{K}$ and for this we need the minimal polynomial of the matrix

$$(4.2.9) \quad \underline{Q} = -\hat{\underline{R}} \underline{B}^{-1} \underline{G}$$

Let this minimal polynomial is

$$m(\lambda) = \alpha_0 + \alpha_1 \lambda + \dots + \alpha_{r-1} \lambda^{r-1} + \lambda^r$$

Knowing $m(\lambda)$ the function containing the same coefficients can be written as:

$$(4.2.10) \quad F(t) = 1 + \alpha_{r-1} t + \dots + \alpha_1 t^{r-1} + \alpha_0 t^r$$

If we introduce the notation

$$F_k = 1 + \alpha_{r-1} t + \alpha_{r-2} t^2 + \dots + \alpha_{k+1} t^{r-k-1} \quad (k=0,1,\dots,r-1)$$

then the inverse of $\underline{K}$ can be given in the form

$$(4.2.11) \quad \underline{K}^{-1} = \frac{1}{F(t)} (\underline{F}_0 t^0 \underline{Q}^0 + \underline{F}_1 t^1 \underline{Q}^1 + \dots + \underline{F}_{r-1} t^{r-1} \underline{Q}^{r-1}) \\ = \frac{1}{F(t)} \sum_{k=0}^{r-1} \underline{F}_k t^k \underline{Q}^k$$

Where by definition

$$t^0 = 1 \quad \text{and} \quad \underline{Q}^0 = \underline{E}$$

Then we introduce the following notations:

$$t = \hat{t} + \varepsilon$$

$$\underline{B}(\hat{t}) = \underline{B}_1$$

$$\underline{L}(\hat{t}) = \underline{L}_1$$

By these

$$\underline{B}(t) = \underline{B}_1 + \varepsilon \hat{\underline{B}} = \underline{B}_1 + \varepsilon \underline{G} \underline{R}$$

and,

$$\underline{L}(t) = \underline{L}_1 + \varepsilon \underline{H} = \underline{L}_1 + \varepsilon \underline{G} \underline{R}$$

So for $t = \hat{t}$, the matrix corresponding to (4.2.6) must be

$$(4.2.13) \quad \begin{bmatrix} \underline{L}_1 & \underline{G} \\ \underline{R} & \underline{O} \end{bmatrix}$$

and this must be examined in the basis belonging to the matrix corresponding to (4.2.7)

$$\begin{bmatrix} \underline{B}_1 & \underline{O} \\ \hat{\underline{R}} & \underline{E} \end{bmatrix}$$

So we get the matrix

$$\hat{\underline{C}}_1 = \begin{bmatrix} \underline{B}_1 & \underline{O} \\ \hat{\underline{R}} & \underline{E} \end{bmatrix}^{-1} \begin{bmatrix} \underline{L}_1 & \underline{G} \\ \underline{R} & \underline{O} \end{bmatrix} = \begin{bmatrix} \underline{B}_1^{-1} \underline{L}_1 & \underline{B}_1^{-1} \underline{G} \\ \underline{R} - \hat{\underline{R}} \underline{B}_1^{-1} \underline{L}_1 & -\hat{\underline{R}} \underline{B}_1^{-1} \underline{G} \end{bmatrix}$$

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Which has the same meaning as the matrix $\hat{C}_1$ defined in the previous point.

Then in connection with the matrix $\hat{C}_1$ we come to the matrix C_2 exactly in the way as in point 4.1.

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