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THEORY OF GAMES

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Contents

1. Historical Background

The modern mathematical approach to interest conflict -game theory - is generally attributed to von Neumann[1] in his papers of 1928 and 1937, although recently Fréchet has raised a question of priority by suggesting that several papers by Borel[2] in the year 1920 really laid the foundations of game theory. These papers have been translated into English and republished with comments by Fréchet and von Neumann.[3] Although Borel gave a clear statement of an important class of game theoretic problems and introduced the concepts of pure and mixed strategies, von Neumann points out that he did not obtain one crucial result - the minimax theorem - without which no theory of games can be said to exist. In fact, Borel conjectured that the minimax theorem is false in general, although he did prove it true in certain special cases. Von Neumann proved it true under general conditions, and in addition he created the conceptually rich theory of

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1) Von Neumann, J.; Zur Theorie der Gesellschaftsspiele, Mathematische annalen, 100, 1928.
Von Neumann, J.; Uber ein ökonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, Ergebnisse eines Mathematik Kolloquiums, 8, 1937.

2) Borel, E; The theory of play and integral equations with skew kernels, and On games that involve chance and the skill of the players; translated by L.J. Savage; Econometrica 21, 1953.

3) Fréchet, M. and J. von Neumann; Commentary on the Borel note, Econometrica, 21, 1953.

games with more than two players.

There were no papers of interest before the publication in 1944 of von Neumann and Morgenstern's book[1] and some mathematical journals. Fortunately, von Neumann and Morgenstern attempted to write their book so that a reader with limited mathematical background could absorb the motivation, the reasoning and the conclusions of the theory. Only a very few scientific volumes as mathematical as this one have published as much interest and general admiration. The recent wars, since that time, were an important contributing factor to the later rapid development of the theory. During that period considerable activity developed in scientific, or at least systematic, approaches to problems that had been previously considered the exclusive province of men of experience. These include such topics as logistics, submarine search, air defence, and other fields. Game theory certainly fits into this trend, and it is the most important theoretical structure that gives very good results.

Although it is not directly relevant to the theory itself, it is worth emphasizing again that game theory is primarily a product of mathematicians and not of scientists from the empirical fields.

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1) The original edition of (Theory of Games and Economic Behavior) appeared in 1944, but the revised edition of 1947 is the more standard reference and it includes the first statement of the theory of utility.

2. Introduction

In linear programming a solution is determined which maximizes one objective function subject to a number of constraints. This technique is applicable in situations in which there is only one decisionmaker or in which the interests of more than one decisionmaker coincide. In real-life situations, however, there are usually a number of decisionmakers with diverging interests. Usually, the interests of decisionmakers are conflicting. Game theory deals with situations with at least two decisionmakers with diverging interests. The word game reflects the fact that most situations of this kind can be imagined in the harmless context of parlour games.

In the general view of conflicts, it would seem that game theory applications should be numerous. The difficulty is that when most realistic cases of this nature are considered, the problems are so complicated that game theory can give few generally valid results. Only in very simple cases a general theory can be developed for situations with conflicting interests. We shall consider in this memo these cases, which are also the best known in game theory.

The kind of games which are considered are the <u>two - person Zero - sum</u> games. These games involve two decisionmakers with diametrically opposed interests. Each may select one alternative out of a finite number of alternatives; the outcome, which is also called pay - off (the value of the objective function for each player), depends on the decisions of both players. Because the game is Zero - sum, the gain of one player is the loss of his opponent and vice virsa.

Since in two - person Zero - sum games the interests of the two sides are totally conflicting, these games should accurately reflect war situations. It is therefore not surprising that most well - known examples of application of game theory deal with armed conflict. A well - known example is the contest between British anti-submarine planes and German submarines in which both the planes and the submarines could have or not have radar equipment. Cold war situations lend themselves rather well to the application of game theory. Companies competing in the same market sometimes are in situations which may be approximated by two - person Zero - sum games.

Game theory as a scientific field can be said to have its origin with the work of von Neumann and Morgenstern (Theory of Games and Economic Behavior). They hoped that the theory of games would play an important role in the explanation of economic behavior. Which is not fulfilled up to now.

In section 3 and explanation of the model of two - person Zero - sum games will be discussed by means of an example. Then it will be shown that for both players decisions can be found which are optimal in a certain sense. The decision problem for both players is formulated as a linear programming problem, which is solved by the simplex method.

3. Example

Consider the following situation (it is based on a note by R.S. Beresford and M.H. Peston[1] describing a situation in Malaya in which game theory was unconsciously applied). A convoy of food trucks travels daily along a road where it is subject to occasional attacks by guerillas. The trucks are protected by an escort of four military vehicles equipped with machine guns and carrying a small force of infantry. The guerillas can pursue two courses of action:

1) They can stage a full - scale attack on the convoy.

2) They may develop sniping activity.

They have to decide which course of action should be followed before the convoy is seen and once they have decided they cannot change their decision.

To counter the guerilla attacks, the four military escort vehicles may be distributed in the convoy in various ways. In any case there should be at least one military vehicle in front and one in the rear, but the other two may be distributed as desired. This

1) Beresford, R.S. and Peston, M.H; A Mixed Strategy in Action, Operations Research Quarterly, Vol 6, 1955.

amounts to the following possibilities:

1) 1 vehicle in front and 3 in the rear,

2) 1 vehicle in front, 1 in the rear and 2 distributed in the convoy.

3) 2 vehicles in front and 2 in the rear.

4) 3 vehicles in front and 1 in the rear.

If the guerillas launch a full - scale attack on the convoy, the escort should preferably be concentrated and so placed that it tends to be carried towards the ambush position rather than away from it. If the guerillas are sniping, it is desirable that the escort should be dispersed as much as possible, so that it can spot where the fire is coming from and return it quickly.

The various alternatives for both sides are called <u>strategies</u>, the escort has 4 strategies and the guerillas have 2. The total number of possible outcomes is therefore 4.2 = 8. We now assume that it is possible to evaluate the gains and losses of the escort for each of the outcomes in a single number, the pay - off. Suppose that these numbers are as in the following Table 1. Hence, if the escort has the (1,3) configuration and a full - scale attack is launched, the gain for the escort is 10, but if the guerillas have decided to snipe, the gain is - 4, and so on.

Players			Guerillas	
	Strategy description		Full-scale attack	Sniping
		Strategy number	1	2
			Pay - Off	
Escorts	1 front, 3 rear	1	10	-4
	Distributed	2	-6	3
	2 front, 2 rear	3	0	-4
	3 front, 1 rear	4	-2	-5

Table 1

In this situation it can be assumed that any gain to the escort is a loss to the guerillas and vice versa; this means that it conforms to a two - person Zero - sum game. The outcomes of the various strategies can be grouped into a rectangular array of numbers, which is called the <u>pay - off</u> matrix of the game. Since the escort wants to maximize the outcome of the game, it is called the <u>maximizing player</u>, its adversary, the guerilla group, is then the <u>minimizing player</u>. It is easy to reverse the role of both players, which is done by multiplication of the pay - off maxtrix by - 1.

What is the best strategy for the escort? If it is known that the guerillas use their first strategy, a full - scale attack, the escort is advised to use its first strategy, thus gaining 10. If it is known that the guerillas use their second strategy sniping, the escort should use its second strategy, there-by gaining 3. But the escort does not know the strategy of the guerillas, so that it cannot, in a straight forward manner, determine its best strategy. The guerillas are in entirely the same position. This is a typical situation in a game.

Let us compare the first and the third strategy of the escort. If the guerillas stage a full - scale attack, strategy 1 has a pay - off of 10 and strategy 3 a pay - off of 0. If the guerillas have decided to snipe, both strategy 1 and strategy 3 have the same pay - off of -4. Since strategy 1 has a higher outcome than strategy 3 if the guerillas use their first strategy and the same outcome if they use their second strategy, it is not rational for the escort to use their third strategy. It is said that the third strategy is dominated by the frist strategy.

In general, a strategy <u>dominates</u> [illegible] dominates another strategy, if for all strategies of the opponent the first strategy has pay - offs which are at least as high as the other strategy, and at least one pay - off which is higher. For example, strategy 4 of the escort is dominated by strategy 1 and also by strategy 3. Since dominated strategies, if players are rational, will never be played, their rows or columns in the pay - off matrix may be deleted, which simplifies the game. In the example, the pay - off matrix is reduced to

$$\begin{array}{cc} 10 & -4 \\ -6 & 3. \end{array}$$

The existence of dominated strategies for one player may lead to dominated strategies for the other player. Let us consider the game with the pay - off matrix given by Table 2. Player A is the maximizing player and player B the minimizing player. When inspecting the pay - off matrix, we find that for A strategy 1 is dominated by strategy 3. If B assumes

Table 2

		B			
		1	2	3	4
A	1	- 2	2	- 1	1
	2	1	4	- 2	- 3
	3	0	2	0	1
	4	- 1	- 2	- 1	5

that A will not play his strategy 1, he can delete the first row of his pay - off matrix. In the resulting matrix, the first strategy of B is dominated by the third one (note that B is the minimizing player). Hence the first column can be deleted. The resulting pay - off matrix is (Table 3)

		B		
		2	3	4
A	2	4	- 2	- 3
	3	2	0	1
	4	- 2	- 1	5

Table 3

There are various ways in which the players can select their strategy. One way is to determine the worst pay - off for each strategy, assuming that the opponent will select the strategy corresponding with this pay - off; then the strategy with the maximum of these minimum pay - offs is chosen. In the reduced pay - off matrix above, the worst pay - off in strategy 2 of A is - 3, that in strategy 3 is 0 and that in strategy 4 is -2. The maximum of these three numbers is 0, which belongs to strategy 3. B can follow the same policy. Taking into account that B is the minimizing player, we find that the worst outcomes for his strategies 2,3 and 4 are 4,0 and 5. The minimum of these is 0 which corresponds to strategy 3.

Now, suppose that both A and B play the strategies determined in this manner, hence both play their strategy 3. Then A will have no occasion to change his strategy, since if he plays his strategy 2 or 4, he will have a pay - off of - 2 or - 1. In the same way we find that B will have no occasion to play anything else than his strategy 3, because if he plays his strategy 2 or 4, the pay - off will be 2 and 1. Once A and B both play their strategy 3, they are not likely to deviate from these. This pair of strategies is called the saddle - point of the game. The saddle - point of a game is a pair of strategies such that there is no lower number in the row of the strategy of the maximizing player and no higher number in the column of the minimizing player. If a game has a saddle - point, it must be found by taking the maximum of the minima of the elements of each row for the maximizing player and the minimum of the maxima of each column for the minimizing player.

Not every game has a saddle - point. Let us return to the escort - guerilla game, for which the pay - off matrix is given in Table 1. The row - minima for the escorts are -4, - 6, - 4 and - 5, the maximum of which is - 4, which belongs to the strategies 1 and 3. The column maxima are 10 and 3; the minimum of these is 3, which belongs to strategy 2. Now the pairs of strategies (1,2) and (3,2) will not be saddle - points. Suppose that at one time the escorts use strategy 1 and the guerillas strategy 2. Then the escorts will next time, assuming that the guerillas will not change their strategy, change over to strategy 2. The next time, the guerillas will, assuming that the escorts will continue to use their strategy 2, change to strategy 1. Then next time the escorts are going to use their strategy 1, and so on .There is no single pair of strategies which both players continue to use because there is no saddle - point. We shall see that also for these games strategies can be found which guarantee each player a certain pay - off.

Instead of deciding to use one strategy, the escorts may decide to use more than one strategy. For instance, they may decide to play their strategy 1 with probability ½ and their strategy 2 with probability ½ . This means that before each game is played, the escorts throw a coin; if heads come up, strategy 1 is used and if tails come up, strategy 2 is played. In this case it is said that a mixed strategy is used. As before it is assumed that at the start of the game the opponent does not know which specific strategy is going to be played by the first player. If the game is played a number of times and the first player is always using the same mixed strategy, the opponent may estimate the probabilities with which the different strategies are being played, but he does not know the strategy to be employed in any single game; until the player of the mixed strategy has thrown the coin or used a similar device, he does not know this himself.

If the escorts are using the mixed strategy of playing the first strategy with probability ½ and the second strategy with probability ½, the average pay - off will be if the guerillas are using their strategy 1.

$$\frac{1}{2} \cdot 10 + \frac{1}{2} \cdot (-6) = 2$$

and if the guerillas use their strategy 2

$$\frac{1}{2} \cdot (-4) + \frac{1}{2} \cdot 3 = -\frac{1}{2} \cdot$$

The summation of the pay - offs times the probabilities implies that the players do not have risk - preference or risk - aversion. If the outcome of a particular game is rather insignificant compared with the total interests of the player, such an opproch is valid. This is the case in the cold war, where the pay - offs in local conflicts are relatively small and where a game may be repeated many times.

4. Game Theory and Linear Programming

If the escorts use the mixed strategy with probabilities of ½ the worst average pay- off which the escorts may expect is — 1/2 , which is found when the guerillas are using their second strategy. This exceeds the minimum pay - off which is obtained by playing their pure strategies 1 or 2, where, if the guerillas accurately guessed the strategy to be played, the pay - offs would be - 4 and - 6. Now let us suppose that the escorts are using their first strategy with probability p_1 and their second strategy with probability p_2, where

$$p_1 + p_2 = 1 \qquad (1)$$

and

$$p_1 \, , \, p_2 \geqslant 0. \qquad (2)$$

In selecting p_1 and p_2 the escorts may use the same objective as before, taking into account that for mixed strategies average pay - offs should be considered. Hence p_1 and p_2 should be determined in such a way that the average pay - off is maximized under the assumption that the opponent uses his best strategy. Let us call this maximum pay - off w. Then, if the guerillas use their first strategy, the average pay - off will be $10\ p_1 - 6\ p_2$ and since w is the pay - off obtained if the opponent uses his best strategy, this average pay - off should be at least equal to w:

$$10\ p_1 - 6\ p_2 \geqslant w \qquad (3)$$

If the guerillas use their second strategy, the pay - off should be at least equal to w, so that the following relation should be valid:

$$-4\ p_1 + 3\ p_2 \geqslant w \qquad (4)$$

The escorts want to maximize w, so that the following objective function may be formulated:

$$f = w \qquad (5)$$

The maximization of f as in (5) subject to the constraints (1) to (4) is a linear programming problem; it may be solved by the simplex method or any other linear programming method. Note that in this problem w has no non - negativity restriction.

Two remarks can be made. In the above, it was assumed that the guerillas used a pure strategy. Is the problem formulation (1) to (5) also valid if the guerillas use a mixed strategy too? Suppose that the guerillas use a mixed strategy with q_1 and q_2 as the probabilities for their first and second strategy. q_1 and q_2 should satisfy

$$q_1 + q_2 = 1 \qquad (6)$$

$$q_1 \, , \, q_2 \geqslant 0 \qquad (7)$$

Then the average pay - off for the escorts will be

$$q_1 \, (10 \, p_1 - 6 \, p_2) + q_2 \, (- 4 \, p_1 + 3 \, p_2)$$

and this should be at least equal to w:

$$q_1 \, (10 \, p_1 - 6p_2) + q_2 \, (- 4p_1 + 3p_2) \geqslant w \quad (8)$$

This inequality can be rewitten by applying (6) as:

$$q_1 \, (10 \, p_1 - 6p_2) + q_2 \, (-4p_1 + 3p_2) \geqslant q_1 w + q_2 \, w \quad (9)$$

If (3) and (4) are satisfied, then, since the q's are non-negative (9) is satisfied. Hence it is sufficient to consider only the pure strategies of the opponent.

What happens if the dominated strategies 3 and 4 of the escorts are also taken into account ? Let us define p_1, p_2, p_3 and p_4 as the probabilities of the mixed strategy of the escorts. The constraints are now

$$10\ p_1 - 6\ p_2 + 0\ p_3 - 2\ p_4 \geqslant w \tag{10}$$

$$-4\ p_1 + 3\ p_2 - 4\ p_3 - 5\ p_4 \geqslant w \tag{11}$$

$$p_1 + p_2 + p_3 + p_4 = 1 \tag{12}$$

$$p_1,\ p_2,\ p_3,\ p_4 \geqslant 0. \tag{13}$$

Now it may be proved that, if a mixed strategy is considered in which a dominated strategy occurs with a positive probability, then the probability of the dominating strategy may be increased at the expense of the dominated strategy, increasing the average pay - off or leaving it the same. For example, consider strategy 4 which is dominated by strategy 1. The terms in p_1 and p_4 in (10) to (12) may be written as

$$12\ p_1 - 2\ (p_1 + p_4)\ , \tag{14}$$

$$p_1 - 5\ (p_1 + p_4)\ , \tag{15}$$

$$(p_1 + p_4)\ . \tag{16}$$

If p_1 is increased and p_4 is decreased by the same amount, $p_1 + p_4$ is unchanged, which means that both (14) and (15) increase; according to (10) and (11), w can be increased by the same amount. Hence no mixed strategy containing a positive value for p_4 can be optinal.

Now the optimal mixed strategy for the escorts will be determined by means of the simplex method. If slack variables are introduced, the problem can be formulated as follows:

Maximize $f = w$ (17)

s.t. $p_1 + p_2 = 1,$ (18)

$10\ p_1 - 6\ p_2 - w - y_1 = 0,$ (19)

$-4\ p_1 + 3\ p_2 - w - y_2 = 0,$ (20)

$p_1, p_2, y_1, y_2 \geq 0$ (21)

Note that w is unrestricted. Artificial variables could be introduced into each of the equations (18), (19) and (20), but (19) and (20) may be multiplied by - 1, after which y_1 and y_2 may be used as basic variables with a nonnegative value:

$$- 10\ p_1 + 6 p_2 + w + y_1 = 0 \quad (22)$$

$$4\ p_1 - 3 p_2 + w + y_2 = 0 \quad (23)$$

In (18), the artificial variable Z may inserted:

$$1 = p_1 + p_2 + Z \quad (24)$$

Z is added to the objective function with the coefficient - M:

$$f = w - M\ Z \quad (25)$$

or

$$0 = - w + M\ Z + f \quad (26)$$

after subtraction of M times (24) from (26) we find

$$- M = - M\ p_1 - M\ p_2 - w + f \quad (27)$$

This can be used for the f_c - and f_m - row of the set - up tableau (Tableau 0 of the following table 4). The equations (22) to (24) furnish the remaining rows.

After two iterations of the simplex method, the Z - variable has disappeared from the basis; the optimal solution is found in one

Tableau	Basic Variables	Values Basic Variables	Ratio	p_1	p_2	w
	f_c	0		0	0	− 1
	f_M	− 1		− 1	− 1	0
0	Z	1	1	1	1	0
	y_1	0		−10	6	1
	y_2	0	0	4	− 3	1
				y_2	p_2	w
	f_c	0		0	0	− 1
	f_M	− 1		1/4	− 7/4	1/4
1	z	1	7/4	− 1/4	7/4	− 1/4
	y_1	0		10/4	− 6/4	14/4
	p_1	0		1/4	− 3/4	1/4
				y_2		w
	f_c	0		0		− 1
	f_M	0		0		0
2	p_2	4/7		− 1/7		− 1/7
	y_1	6/7	6/23	16/7		23/7
	p_1	3/7	3	1/7		1/7
				y_2		y_1
	f	6/23		16/23		7/23
3	p_2	14/23		− 1/23		1/23
	w	6/23		16/23		7/23
	p_1	9/23		1/23		− 1/23

Table (4)
(Application of the simplex method to the problem of the maximizing player)

further iteration. Note that as long as Z is a basic variable, the f_m - row is identical to minus the Z - row; hence the f_m - row never needs to be written down. Furthermore, as long as w does not enter the basis, the f_c - row consists of zeroes except for the element in the w - column. After w has entered the basis, the f_c - row is indential to the w - row. Hence also the f_c - row can be deleted altogether.

$$f = \frac{6}{23}$$

$$p_1 = \frac{9}{23}$$

$$p_2 = \frac{14}{23}$$

This means that the escorts should, each time before the convoy starts, pick at random a number $1 \rightarrow 23$; if it is $1 \rightarrow 9$, the first strategy should be used, if it is $10 \rightarrow 23$, the second strategy should be employed. The average pay - off is then at least 6/23. Of course, if the guerillas are using their first strategy consistently, it is better for the escorts to use their first strategy; if the guerillas use their second strategy consistently, the escorts should use their second strategy. But if nothing is known about the strategies of the opponent, the solution outlined guarantees the escorts a minimum payoff of $\frac{6}{23}$.

The guerillas face a similar problem. They too can use a mixed strategy; let the probabilities associated with their first and second strategy he q_1 and q_2. The guerillas want to minimize the maximum pay-off as it is given by the pay-off matrix. Let the maximum pay-off be v. Then if the escorts use their first strategy, the pay-off would be $10q_1 - 4q_2$, and this should at most equal.

$$10q_1 - 4q_2 \leq v \tag{28}$$

If the escorts use their second strategy, the pay-off would be $-6q_1 + 3q_2$ and this should at most equal v :

$$-6q_1 + 3q_2 \leq v \tag{29}$$

The probabilities q_1 and q_2 should be nonnegative and add to 1:

$$q_1 \geq 0 , \quad q_2 \geq 0, \tag{30}$$

$$q_1 + q_2 = 1 \tag{31}$$

The maximum pay-off should be minimized, so that the objective function is :

minimize $\quad g = v$

Hence the complete problem for the minimizing player is :

$$\text{Minimize} \quad g = v \tag{32}$$

$$\left.\begin{aligned} \text{Subject to} \quad 10q_1 - 4q_2 - v &\leq 0 \\ -6q_1 + 3q_2 - v &\leq 0 \\ q_1 + q_2 &= 1 \\ q_1 \geq 0 , \; q_2 &\geq 0 \end{aligned}\right\} \tag{33}$$

The complete problem for the maximizing player was :

$$\text{Maximize} \quad f = w \tag{34}$$

$$\text{Subject to} \quad \left.\begin{array}{l} 10p_1 - 6p_2 - w \geqslant 0 \\ -4p_1 + 3\,p_2 - w \geqslant 0 \\ p_1 + p_2 \qquad\quad = 1 \\ p_1 \geqslant 0 \ , \ p_2 \geqslant 0 \end{array}\right\} \qquad (35)$$

It can be proved that the minimization problem is the dual problem of the maximizatioh problem and vice versa (see excercise 2). Hence the values of the basic variables of the dual problem should appear as the elements of the f-row in the primal problem. The optimall solution for the problem of the guerillas is therefore

$q = 6/23$,

$q_1 = 7/23$,

$q_2 = 16/23$

The guerillas should , before they start out, tahe a random number 1-23 , if it is 1-7 , they should launch a full-scale attack, if it is 8-23 they should display sniping activity . Neither party would in this situation be able to find a higher (for the escorts) or lower (for the guerillas) average pay-off by changing their mixed strategy; in fact, it they did, the opponent could take advantage of it and have a better average pay-off.

Exercises :

1. Two companies share a certain market. Each company can in any year follow two marketing strategies; it can advertise or give price discounts. In terms of market share for the first company the outcomes are as follows:

		B	
		Advertise	Price discount
A	advertise	70	50
	Price discount	55	60

What are the optimal strategies for both companies if both companies maximize market share ? What is the corresponding average market share of company A ?

2. Given is a two-person zero-sum game in which the maximizing player A has n_A strategies and the minimizing player B has n_B strategies. The pay-off for A if the plays his i th strategy and B plays his j th strategy is a_{ij} . Formulate the problem of finding the optimal strategy for A as a linear programming problem. Do the same for B. show that both problems are each other's dual problem.

3. Analyze the game with the following general pay-off matrix.

		B	
		1	2
A	1	a	b
	2	c	d

for all values of a, b, c, and d.

<u>For further readings</u>:

1. Bennion, E.G.; Linear Programming and Game Theory, MSU Business Studies, 1960.

2. Blackwell, D and Girshick, M.A.; Theory of Games and Statistical Decisions, John Wiley & sons, N.Y. 1954.

3. Kuhn, H.W. and Tucker, A.W. (ed.); Contributions to the Theory of Games, Annals of Mathematics Studies, Number 28, Princetion New Jersey, P.U.P, 1953. Vol I and Vol. II.

4. Luce, D. and Raiffa, H.; Games and Decisions, John Wiley and Sons, N.Y. 1958.

5. Vajda, S.; The Theory of Games and Linear Programming, Methuen & Co, London, 1961.