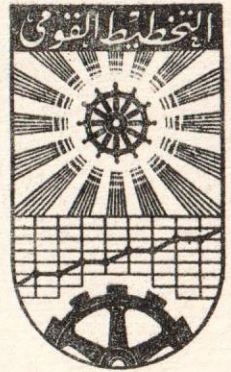


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Lecture notes in Operations Research
RELATED TOPICS TO
LINEAR PROGRAMMING

By

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CHAPTER 1

The Transportation Problem

1. The Transportation Method Of Linear Programming

The transportation problem is a very special case of linear programming. It is of particular interest because of its applicability in economics and of its simplicity in computations. The transportation problem is originated and solved by F.L. Hitchcock several years before the general concept of linear programming was formulated. This problem has numerous industrial and economic applications which have nothing to do with transportation, but the name, which it took from its original formulation, is still used. The most remarkable characteristic of this problem is that whereas linear programming problems require high speed computers, the transportation problem is frequently best solved manually.

To be familiar with the problem, it is better to introduce the following example.

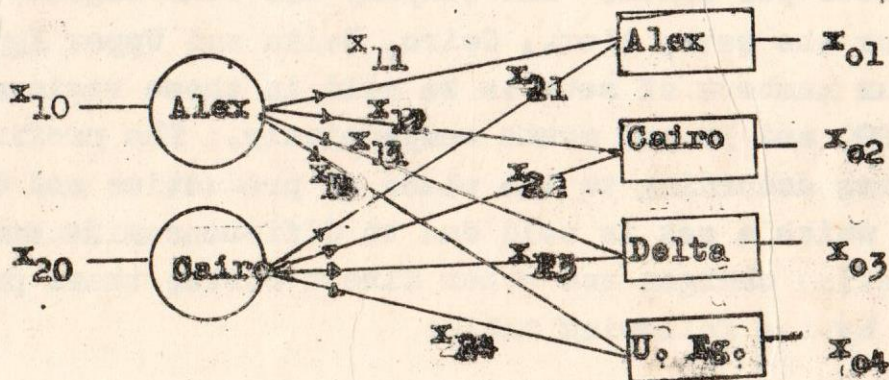
Let us consider the following problem. The ELNASR T.V. Company has two plants, one in Alexandria with a capacity of 100 T.V. sets per month and one in Cairo with a capacity of 200 T.V. sets per month. The company has four regions as its markets for the sets, Alex., Cairo, Delta and Upper Egypt; the maximum numbers of sets to be sold in these regions are 75, 100, 200 and 30 per month respectively. The profit per unit differs according to the place of production and the region in which a set is sold due to differences in costs, transportation charges and other direct costs; these profits are given by the following table:

Profits per unit (per piaster)

		Region of Sales			
		Alex.	Cairo	Delta	Upper Egypt
place of production	Alex.	80	0	20	20
	Cairo	116	114	76	148

It is assumed that only variable costs are calculated. The company may leave its productive capacity at both plants partly or wholly unused and it may leave demand in any country unsatisfied without any costs. The problem now is: How should it allocate the productive capacity of the two plants over the four markets in order to maximize total profits on T.V. sets?

A problem of such a small size can easily be solved by checking all possible allocations or just those that look promising. Even for problems that are only slightly larger this is not practical because of the number of different allocations that are possible. It is therefore necessary to develop a more systematic procedure.



First, the problem will be formulated in terms of equations. Let us indicate the number of sets produced by the Alex. plant for the Alex. market with x_{11} , for the Cairo market by x_{12} , for the Delta market by x_{13} and for the Upper Egypt market by x_{14} . Similarly for the Cairo plant by x_{21} , x_{22} , x_{23} and x_{24} respectively. Further, let the unused capacity of the Alex. plant be x_{10} and the unused capacity of the Cairo plant x_{20} . Since total production of a plant plus its unused capacity must be equal to its capacity, the following equations are valid:

$$x_{10} + \sum_{j=1}^4 x_{1j} = 100, \quad (1)_a$$

$$x_{20} + \sum_{j=1}^4 x_{2j} = 200. \quad (1)_b$$

Also, let us denote the unsatisfied demands on the four markets by x_{01} , x_{02} , x_{03} and x_{04} , respectively. Since in each region sales from both sources plus unsatisfied demand must equal maximum demand, the following equations are valid:

$$x_{01} + \sum_{i=1}^2 x_{i1} = 75 \quad (2)_a$$

$$x_{02} + \sum_{i=1}^2 x_{i2} = 100 \quad (2)_b$$

$$x_{03} + \sum_{i=1}^2 x_{i3} = 100 \quad (2)_c$$

$$x_{04} + \sum_{i=1}^2 x_{i4} = 30 \quad (2)_d$$

Let us add the first set of equations, we get

$$x_{00} + \sum_{i=1}^2 x_{i0} = 300 \quad (1)_c$$

Where

$$x_{00} = \sum_{i=1}^2 \sum_{j=1}^4 x_{ij}$$

In the same way, the second set of equations can be added, which result in

$$x_{00} + \sum_{j=1}^4 x_{0j} = 305 \quad (2)_e$$

Since the last equation can be obtained from the two sets of equations, one of the equations of the whole equation system is dependent on the other equations.

The equation system can be put into a particularly convenient form by arranging all x-variables in a rectangular array or tableau:

x_{00}	x_{01}	x_{02}	x_{03}	x_{04}
x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
x_{20}	x_{21}	x_{22}	x_{23}	x_{24}

The last equation $(2)_e$ with the first set of equations $(1)_a, (1)_b$ state that the sum of the elements of the three rows should be constant and equal to 305, 100 and 200, resp. The equations $(1)_c$ and $(2)_a, \dots, (2)_d$ state that the sum of the elements of the five columns should be constant and equal to 300, 75, 100, 100 and 30, resp. This representation of the equations simplifies the solution of the problem to a great extent.

As a starting point of the method, we consider the particular solution in which the entire productive capacity is left unused and in which no sales are made in any market. This means:

$$\begin{aligned} x_{10} &= 100, & x_{20} &= 200, \\ x_{01} &= 75, & x_{02} &= 100, & x_{03} &= 100, & x_{04} &= 30; \end{aligned} \quad (3)$$

all other variables are equal to zero. This solution can be put into tableau form as follows:

0	75	100	100	30	
100	0	0	0	0	
200	0	0	0	0	(4)

As defined in memo 488, we distinguish in this solution as well as in the solutions we shall consider later, two kinds of variables, firstly basic variables which may have non-zero values and non-basic variables which only have zero values. In (4) the basic variables are those with nonzero values given in (3) and x_{00} which happens to be zero; the nonbasic variables are all variables not in the first column and in the first row.

In any basic solution, the number of basic variables is equal to the number of rows plus the number of columns minus one, which is equal to the number of independent equations on which the tableau was based.

Now, let us consider the total profits, which are equal to the sum of profits per unit times the number of units sold over the various regions and plants. Hence, if Z is total profits, we have the objective function

$$Z = 80 x_{11} + 0 x_{12} + 20 x_{13} + 20 x_{14} + 116 x_{21} + 144 x_{22} + 76 x_{23} + 148 x_{24} \quad (5)$$

This function may be indicated in a more convenient manner by only writing down the coefficients in a rectangular tableau, since the meaning of the coefficients will be clear from their position, this is

0	0	0	0	0
0	80	0	20	20
0	116	144	76	148

(6)

If the solution given in (4) is substituted in the objective function (5), the value of Z is zero, since all basic variables of (4) have zero coefficients in (5); the nonbasic variables are zero, so that terms in nonbasic variables cannot contribute anything to the objective function (i.e. the solution at the origin). Proceeding from (4), an improved solution will be sought which is done by increasing one of the nonbasic variables. Let us take the nonbasic variables

les x_{24} .

If $x_{24} = \Delta$ there should be changes in the basic variables of solution (4) in order to keep the sums of the elements of rows and columns constant. Since the sum of the elements of the last column of (4) should remain constant, we should replace $x_{04} = 30$ by $x_{04} = 30 - \Delta$. Then, to keep the sum of the elements in the first row constant we should have instead of $x_{00} = 0$, $x_{00} = \Delta$. Since the sum in the first column should be kept constant, we change $x_{20} = 200$ into $x_{20} = 200 - \Delta$; by the last change the sum in the last row is also constant. The following solution is obtained

Δ	75	100	100	$30 - \Delta$	
100	0	0	0	0	(7)
$200 - \Delta$	0	0	0	Δ	

We have introduced a nonbasic variable at a nonzero value Δ and have determined the changes necessary in the basic variables in order that the original set of equations (1) and (2) are still satisfied. The effect on the objective function of this change is only reflected in the coefficient x_{24} , i.e. 148, since all basic variables have zero values in the objective function. For any nonbasic variable of (4), the same reasoning is valid. In general, introduction of a nonbasic variable at a nonzero value

Δ causes changes in the basic variables; the effect on the objective function of the changing solution is given by the coefficient of the nonbasic variable only since the basic variables have zero coefficients. Let us therefore select as the variable to be increased the nonbasic variable with the largest positive coefficient in the objective function.

According to tableau (6) this is x_{24} .

Tableau (7) gives the changed solution in terms of the value of x_{24} i.e. Δ . The value of f is for this solution 148Δ and since profits should be maximized, Δ should be made as large as possible, without, however, making any variable of the solution negative. From (7) it is obvious that the maximum value of Δ is 30, for which value we find for the solution

30	75	100	100	0	
100	0	0	0	0	(8)
170	0	0	0	30.	

We now have obtained another basic solution in which x_{24} in a basic variable which replaced x_{04} (x_{04} now is a nonbasic variable).

This solution should be improved in the same way as the previous solution (4) was improved. To find an improved solution for (4), the objective function (5) was used which had zero coefficients for the basic variables of (4). For the new solution in (8), however, the objective function (5) has a coefficient 148 for the basic variable x_{24} . It is therefore desirable to change the objective function in such a way that it has a zero coefficient for x_{24} ; the coefficient for x_{04} may become nonzero since x_{04} is not a basic variable in the new solution. This change in the objective function may be achieved by subtracting 148 times equation (2)_a, i.e.

$$x_{04} + x_{14} + x_{24} = 30$$

from the objective function (5). The result is

$$\begin{aligned} z - 148.30 = & 0 x_{00} + 0 x_{01} + 0 x_{02} + 0 x_{03} - 148 x_{04} \\ & + 0 x_{10} + 80 x_{11} + 0 x_{12} + 20 x_{13} - 128 x_{14} \\ & + 0 x_{20} + 116 x_{21} + 144 x_{22} + 76 x_{23} + 0 x_{24} \end{aligned}$$

It is obvious that this amounts to subtracting 148 from the elements of the last column of (5), so that we have in tableau form

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & -148 \\ 0 & 80 & 0 & 20 & -128 \\ 0 & 116 & 144 & 76 & 0 \end{array} \quad (9)$$

This form of the objective function has the property that it has zero for the basic variables of the solution in (8), so that it can be used for improving this solution.

Since, we wish to improve the solution, we select the nonbasic variable with the largest positive coefficient in (9), which is x_{22} with a coefficient of 144.

Putting $x_{22} = \Delta$ and adjusting the other basic variables to keep the sums of elements of rows and columns constant from (8) we find

$$\begin{array}{ccccc} 30 + \Delta & 75 & 100 - \Delta & 100 & 0 \\ 100 & 0 & 0 & 0 & 0 \\ 170 - \Delta & 0 & \Delta & 0 & 30 \end{array} \quad (10)$$

The maximum value which Δ can take without making any basic variable negative is 100, so that the next solution is:

$$\begin{array}{ccccc} 130 & 75 & 0 & 100 & 0 \\ 100 & 0 & 0 & 0 & 0 \\ 70 & 0 & 100 & 0 & 30 \end{array} \quad (11)$$

In this solution x_{22} is a basic variable, which replaced x_{02} (x_{02} now is a nonbasic variable).

The adjustment in the objective function necessary to make the coefficient of x_{22} zero is obtained by subtracting 144 times equation (2)_b, i.e.:

$$x_{02} + x_{12} + x_{22} = 100$$

from the objective function implied by (9). The result is

0	0	-144	0	-148	
0	80	-144	20	-128	(12)
0	116	0	76	0	

The left hand side of the equation of the objective function is now

$$Z = 148 \times 30 - 144 \times 100 = Z = 18.840$$

which implies that, since, as before, the right hand side is zero for the basic solution of (11) that $Z = 18.840$.

Next, the solution (11) should be improved by selecting the nonbasic variable with the largest positive coefficient in the corresponding form of the objective given by (12). Hence we put $x_{21} = \Delta$; after adjusting the basic variables for this we find

$130 + \Delta$	$75 - \Delta$	0	100	0	
100	0	0	0	0	(13)
$70 - \Delta$	Δ	100	0	30	

The maximum value of Δ is found to be 70, for this value the solution is:

$$\begin{array}{ccccc}
 200 & 5 & 0 & 100 & 0 \\
 100 & 0 & 0 & 0 & 0 \\
 0 & 70 & 100 & 0 & 30
 \end{array} \quad (14)$$

In this solution x_{21} has replaced x_{20} as a basic variable.

Now, the objective function should be adjusted. In order to reduce the coefficient of x_{21} to zero, 116 times equation (1)_b is subtracted from the objective function having the coefficients given in (12). The resulting coefficients are

$$\begin{array}{ccccc}
 0 & 0 & -144 & 0 & -148 \\
 0 & 80 & -144 & 20 & -128 \\
 -116 & 0 & -116 & -40 & -116
 \end{array} \quad (15)$$

In this tableau x_{21} has a zero coefficient, but now the coefficients of the basic variables x_{22} and x_{24} are nonzero. Hence 116 times the equations (2)_a and (2)_c are added to the objective function equation which results in an equation having as coefficients

$$\begin{array}{ccccc}
 0 & 0 & -28 & 0 & -32 \\
 0 & 80 & -28 & 20 & -12 \\
 -116 & 0 & 0 & -40 & 0
 \end{array} \quad (16)$$

For simplicity, we perform the adjustment of the coefficients of the objective function in the following way: Tableau (12) is the tableau to be transformed. Let us circle the coefficients of its basic variables, which are x_{00} , x_{01} , x_{03} , x_{10} , x_{21} , x_{22} and x_{24} . Subtracting 116 from the last row is indicated by a minus sign within brackets in front of the elements in the last row. It is then observed that the circled coeffi-

Hence 116 is added to the third and fifth columns which is indicated by a plus within brackets in front of the elements concerned. The following tableau is obtained:

$$\begin{array}{ccccc}
 \textcircled{0} & \textcircled{0} & (+)-144 & \textcircled{0} & (+)-148 \\
 \textcircled{0} & 80 & (+)-144 & 20 & (+)-128 \\
 (-)0 & (-)\textcircled{116} & (+)\textcircled{0} & (-)76 & (+)\textcircled{0}
 \end{array} \quad (17)$$

It is observed that now all basic variables have zero coefficients; so that the indicated subtractions and additions may be performed, which results in (16).

The problem can be solved in this manner, but before any further steps are taken a more convenient table will be indicated. It was pointed out in memo 488 that the basic variables of a solution always have zero coefficients in the corresponding formulation of the objective function. On the other hand, nonbasic variables which may have nonzero coefficients have zero values in any basic solution. It is therefore possible to combine in one tableau values of basic variables and coefficients of nonbasic variables; an element of such a tableau is the value of a variable if this variable is basic (the values of basic variables are circled) and is a coefficient of the objective function if the variable is nonbasic. Tableau 0 of the following table combines in this way the tableaux (4) and (6), Tableau 1 combines the tableaux (8) and (9). Tableau 2 combines the tableaux (11) and (12) and Tableau 3 combines the tableaux (14) and (16). The Δ 's within brackets behind an element refer to the new basic variable and the corresponding changes in the basic variables of the current solution; the plus and minus signs within brackets in front of an element have been used for the adjustment of the coefficients of the objective function as described before.

Tableau		Unused Cap.	Alex	Cairo	Delta	U. Egypt
0	uns. Dem.	(0) (Δ)	(75)	(100)	(100)	(-) (30) (-)
	Alex	(100)	80	0	20	(-)
	Cai	(200) (- Δ)	116	144	76	(-) 148 (Δ)
1	uns. Dem.	(30) (Δ)	(75)	(-) (10) (- Δ)	(100)	- 148
	Alex	(100)	80	(-) 0	20	- 128
	Cai	(170) (- Δ)	116	(-) 144 (Δ)	76	(30)
2	uns. Dem.	(30) (Δ)	(75) (- Δ)	(+) -144	(100)	(+) -148
	Alex	(100)	80	(+) -144	20	(+) -128
	Cai	(-) (70) (- Δ)	(-) 116 (Δ)	(+) (100)	(-) 76	(+) (30)
3	uns. Dem.	(200) (Δ)	(-) (5) (- Δ)	(-) -28	(100)	(-) -32
	Alex	(100) (- Δ)	(-) 80 (Δ)	(-) -28	20	(-) -12
	Cai	(+) -116	(+) (70)	(+) (100)	(+) -40	(+) (30)
4	uns. Dem.	(205) (Δ)	- 80	(+) -108	(100) (- Δ)	(+) -112
	Alex	(95) (- Δ)	(5) (Δ)	(+) -108	20	(+) -92
	Cai	(-) -36	(-) (70) (- Δ)	(+) (100)	(-) 40	(+) (30)
5	uns. Dem.	(275) (Δ)	(+) 80	- 68	(30) (-)	- 72
	Alex	(-) (25) (- Δ)	(+) (75)	(-) -68	(-) 20 (Δ)	(-) -52
	Cai	- 76	(+) -40	(100)	(70)	(30)
6	uns. Dem.	(300)	-60	- 68	(5)	- 72
	Alex	- 20	(75)	- 88	(25)	- 72
	Cai	- 76	-20	(100)	(70)	(30)

We shall proceed using these combined tableaux. In Tableau 3, the largest positive element which is not circled is 80, corresponding with x_{11} . Hence x_{11} is the new basic variable, after the adjustments in the basic variables of the current solution, it is found that x_{01} leaves the basis and that $\Delta = 5$. The values of the basic variables of Tableau 4 may be written down accordingly. The coefficients of the objective function are transformed by subtractions and additions of the

coefficient of the new basic variable, 80. First a minus is put in front of the elements of the second column. The first element, 5, of this column is circled, but the basic variable concerned, x_{01} , is becoming nonbasic. The third element of this column, 70, is circled, so that it refers to a value of a basic variable. Hence a plus is put in front of the elements in the third row. Now, two circled elements, 100 and 30, have a sign which should be neutralised, this is done by putting minus signs in front of the elements in the third and fifth columns. The adjustment can be performed as usual.

In Tableau 4 the largest positive element which is not circled is 40, so that x_{23} should become basic. The changes in the basic variables are found in the same way as before. The leaving basic variable turns out to be x_{22} and $\Delta = 70$. In Tableau 5, x_{13} is the new basic variable and x_{10} is the leaving basic variable; Tableau 6 is then found directly.

In Tableau 6, there is no nonbasic variable with a positive coefficient so that it does not pay to increase any nonbasic variable; since the coefficients of nonbasic variables are all negative, any increase in value of these variables, each one separately or any number of them in linear combinations would decrease the objective function. All nonbasic variables are zeros. Hence the values of basic variables given by the basic solution of Tableau 6 constitute the optimal solution. This solution is :

$$x_{00} = 300, \quad x_{03} = 50$$

$$x_{11} = 75, \quad x_{13} = 25.$$

$$x_{22} = 100, \quad x_{23} = 70, \quad x_{24} = 30.$$

This means that, the production of Alex. plant should therefore assigned to Alex. and Delta markets, and the production of Cairo plant should be assigned to Cairo, Delta and Upper Egypt markets. There is an unsatisfied demand on the Delta market of 5.

In the optimal tableau, the coefficients of the objective function are important as the values of the basic variables. For example, the coefficient of x_{21} , which is -20, can be interpreted as the loss that would happen per unit if Cairo - made sets to be sold in Alexandria. The coefficients of x_{10} and x_{20} can be interpreted in the same manner; they are equal to the loss of leaving a unit of productive capacity unused in Alex. and Cairo. In a similar way, the coefficients of x_{01} , x_{02} , x_{03} and x_{04} can be interpreted as the loss incurred if a unit of Alex, Cairo, Delta or Upper Egypt demand is left unsatisfied or as the profit to be made on additional units of demand.

The value of the objective function may either be computed at each step of the method by adding the increase or it may be computed for the optimal solution. The last computation can be performed in two ways. Firstly, the values of the basic variables may be multiplied with their coefficients and the resulting products added. The value of the objective function for the solution of Tableau 6 is then found as

follows:

$$F = 80.75 + 20.25 + 144.100 + 76.70 + 148.30 \\ = 30.660$$

Alternatively, we may multiply the available capacity of each plant and the maximum demand in each country by the values that can be ascribed to them which are given by the coefficients in the optimal tableau of the variables x_{10} , x_{20} , x_{01} , x_{02} , x_{03} , and x_{04} . For our example this is

$$F = 20.100 + 76.200 + 60.75 + 68.100 + 0.100 + 72.30 \\ = 30.660$$

The method which has been described is usually called the transportation method of linear programming. Sometimes, it is called the simplex transportation method.

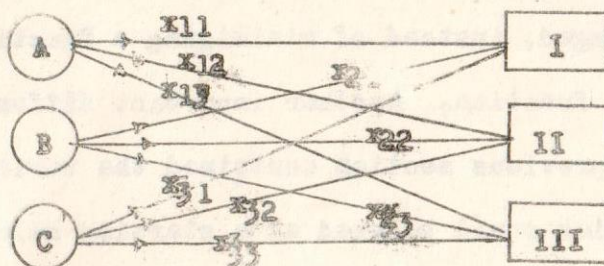
2. The Classical Transportation Problem

A company has plants in A, B and C and warehouses in I, II and III. In A, B and C it has 4, 4 and 2 units available of a certain commodity, while in I, II and III, 1, 6 and 3 units of this commodity are required. The transport costs per unit from any plant to any warehouse are given in the following table:

		Warehouses			Availabilities
		I	II	III	
Plants	A	6	8	3	4
	B	2	3	1	4
	C	2	5	6	2
		(Costs per unit)			
Requirements		1	6	3	

Which transportation program minimizes total transportation costs?

This problem may be formulated in a mathematical model as follows.



Let x_{11} be the quantity transported from A to I; x_{12} the quantity transported from A to II; and so on. The problem is then to minimize total transport costs

$$\begin{aligned} Z = & 6x_{11} + 8x_{12} + 3x_{13} \\ & + 2x_{21} + 3x_{22} + x_{23} \\ & + 2x_{31} + 5x_{32} + 6x_{33} \end{aligned} \quad (1)$$

The solution should satisfy the constraints

$$\sum_{i=1}^3 x_{1i} = 4 \quad (2)_a$$

$$\sum_{i=1}^3 x_{2i} = 4 \quad (2)_b$$

$$\sum_{i=1}^3 x_{3i} = 2 \quad (2)_c$$

$$\sum_{j=1}^3 x_{1j} = 1 \quad (2)_d$$

$$\sum_{j=1}^3 x_{2j} = 6 \quad (2)_e$$

$$\sum_{j=1}^3 x_{3j} = 3 \quad (2)_f$$

and $x_{ij} \geq 0 \quad (i = 1, 2, 3; j = 1, 2, 3) \quad (2)_g$

In the problem of the previous section, we maximized the objective function whereas it is minimized here. This is a difference which is easily changed, instead of minimizing a function we may maximize minus that function. Another important difference is that the problem of the previous section contained the variables x_{00} , x_{01} , ..., x_{30} which could be used as a starting solution of the problem because their coefficients in the objective function were zero. In the classical transportation problem no such obvious starting solution is available.

The variables x_{00} , x_{01} , ..., x_{30} do not occur in the problem as formulated in (1) and (2) but we may include them in the formulation as artificial variables, except for x_{00} which is nonnegative. Hence the problem may be reformulated as follows:

$$\begin{aligned} \text{Minimize } f = & 0x_{00} + 0x_{01} + 0x_{02} + 0x_{03} \\ & + 0x_{10} + 6x_{11} + 8x_{12} + 3x_{13} \\ & + 0x_{20} + 2x_{21} + 3x_{22} + x_{23} \\ & + 0x_{30} + 2x_{31} + 5x_{32} + 6x_{33} \end{aligned} \quad (3)$$

subject to the constraints

$$x_{00} + x_{01} + x_{02} + x_{03} = 10, \quad (4)a$$

$$x_{10} + x_{11} + x_{12} + x_{13} = 4, \quad (4)b$$

$$x_{20} + x_{21} + x_{22} + x_{23} = 4, \quad (4)c$$

$$x_{30} + x_{31} + x_{32} + x_{33} = 2, \quad (4)d$$

$$x_{00} + x_{10} + x_{20} + x_{30} = 10, \quad (4)e$$

$$x_{01} + x_{11} + x_{21} + x_{31} = 1, \quad (4)f$$

$$x_{02} + x_{12} + x_{22} + x_{32} = 6, \quad (4)g$$

$$x_{03} + x_{13} + x_{23} + x_{33} = 3, \quad (4)h$$

$$x_{01} + x_{02} + x_{03} + x_{10} + x_{20} + x_{30} = 0 \quad (4)k$$

All variables should be nonnegative. The constraints (4)a and (4)e are added as explained in Section 1. Except minimizing the objective function (instead of maximization) and the constraint (4)a, the formulation of the problem is the same as in the previous section. Hence, we can put the problem in a transportation tableau format, circling basic variables. Tableau 0 of the following table gives this tableau in which the variables given in (4)k are basic.

Tableau		0	I	II	III
0	O	10	1	6	3
	A	4	6(1)	8(3)	3
	B	4	2	3(3)	1(1)
	C	2	2	5	6(2)
1	O	10	-6	-8	-6
	A	0	1(-A)	3(A)	-3
	B	5	1	3(-A)	1(A)
	C	0	(+)-4(A)	-3	2(-A)
2	O	10	(+)-2	-8	(+)-6
	A	0	(+) 4	4(-A)	(+)-3(A)
	B	5	(+) 5	2(A)	(+)-2(-A)
	C	(-) 0	(+) 1	(-)-3	(+) 1
3	O	10	(-) 1	-8	-3
	A	0	(-) 7	2(-A)	2(A)
	B	5	(+) 8	4	3
	C	(+)-3	(+) 1	(+)-6(A)	(+)-1(-A)
4	O	10	-5	-8	-3
	A	0	1	1	5
	B	5	2	4	3
	C	3	1	1	6

The solution of Tableau 0 obviously does not satisfy (4)_k. Since the simplex transportation method needs a feasible solution to start with, first a feasible solution must be generated. This may be done using the north-west corner rule which is explained as follows. The values of basic variables according to Tableau 0 are

①	①	⑥	③
④	0	0	0
④	0	0	0
②	0	0	0

(5)

Now, a solution should be generated in which no variable in row and column 0 except x_{00} has a nonzero value. Beginning in the north - west corner of that part of the tableau where basic variables are allowed, we assign as many units as possible to x_{11} . The following solution results from this

①	0	⑥	③
③	①	0	0
④	0	0	0
②	0	0	0

(6)

In the usual procedure is followed, now the coefficients of the objective function should be adjusted. However, it is in this case more efficient first to generate a feasible solution and then adjust the objective function accordingly in one step.

Since in (6) the quantity required in (I) is exhausted, but the available quantity in A is not, x_{12} should be made as large as possible. The result is

④	0	③	③
0	①	③	0
④	0	0	0
②	0	0	0

(7)

Now, the quantity available at A is exhausted. Since we started to reduce the required quantity in (II), we go on with this by making x_{22} as large as possible. The result is

(7)	0	0	(3)
0	(1)	(3)	0
(1)	0	(3)	0
(2)	0	0	0

(8)

The required quantity in (II) is exhausted but the quantity available in B is not, so that x_{23} is made as large as possible, the result is

(8)	0	0	(2)
0	(1)	(3)	0
0	0	(3)	(1)
(2)	0	0	0

(9)

Since now the required quantity in (III) should be reduced, the value of x_{33} is made as large as possible. The result is

(10)	0	0	0
0	(1)	(3)	0
0	0	(3)	(1)
(0)	0	0	(2)

(10)

In the last step the new basic variable was x_{33} . As the basic variable becoming nonbasic we may select either x_{03} or x_{30} . This means that x_{30} or x_{03} remains a basic variable but has a value zero.

After obtaining a basic feasible solution to the problem we now move to the adjustment of the objective function. The coefficients we have initially are

0	0	0	0
(0)	(6)	(8)	3
0	2	(3)	(1)
0	2	5	(6)

(11)

Now, the coefficients of basic variables are circled. x_{10} is chosen as a basic variable with value zero instead of x_{30} or x_{03} . From each row or column a quantity should be subtracted in such a way that the circled elements are zero; the amount to be subtracted will be the coefficient of x_{20} , x_{30} , x_{01} , x_{02} and x_{03} with a minus sign. Disregarding the other tableau elements for a moment, and reducing the coefficient of x_{11} to zero, we get.

		-6	0	0
0		0	8	
0			3	1
0				6

(12)

Subtracting 8 from column 2, we find

		-6	-8	0
0		0	0	
0			-5	1
0				6

(13)

The coefficient of x_{22} is reduced to zero by adding 5 to the elements of row 2, which results in

		-6	-8	0
0		0	0	
5			0	6
0				6

(14)

In order to reduce the coefficients of x_{23} to zero, we subtract 6 from the elements in column 3. This results in

		-6	-8	-6
0		0	0	
5			0	0
0				0

(15)

All basic variables having zero coefficients, we now determine the other coefficients by adding the marginal number of each row or column of (15) to the elements of each row and columns of (11). The result is

$$\begin{array}{c|ccc} 0 & -6 & -8 & -6 \\ \hline 0 & 6+0-6 & 8+0-8 & 3+0-6 \\ 5 & 2+5-6 & 3+5-8 & 1+5-6 \\ 0 & 2+0-6 & 5+0-8 & 6+0-6 \end{array} = \begin{array}{c|ccc} 0 & -6 & -8 & -6 \\ \hline 0 & 0 & 0 & -3 \\ 5 & 1 & 0 & 0 \\ 0 & -4 & -3 & 0 \end{array} \quad (16)$$

These are the cost coefficients corresponding with the basic solution of (10) in which x_{10} is taken as a basic variable, (10) and (16) can be combined to give Tableau 1 of the table.

Now, we can proceed as indicated in the last section, noticing that we do not take into the basis any of the variables x_{01} , x_{02} , x_{03} , x_{20} and x_{30} ; further it should be remembered that the objective function is minimized instead of maximized, so that the nonbasic variable with the most negative coefficient should be selected as the new basic variable. Hence x_{31} , which has a coefficient of -4 , is selected. After the necessary adjustments are made for the variables of the current basic solution, x_{11} is found as the basic variable becoming nonbasic and is put at 1. The coefficients of the objective function are adjusted by adding 4 to the coefficients of column 1.

The following two steps are similar and result in Tableau 3 and Tableau 4. The solution of Tableau 4 is optimal since none of the coefficients of nonbasic variables is negative, which means that no decrease in the objective function can take place.

Again, the coefficients of nonbasic variables can be interpreted as the loss that would happened if in deviation from the optimal solution the nonbasic variable would be given a unit value. The coefficients of x_{10} , x_{20} and x_{30} can be interpreted as minus the costs of any

additional unit available in A, B and C and the coefficients of x_{01} , x_{02} and x_{03} as minus the costs of any additional unit required in I, II and III,

Exercise 1:

The Shell company has refineries in Rotterdam, Southampton and Yokohama; the monthly input of crude oil of these refineries is 15, 10 and 10 thousand ton, respectively. The crude oil is supplied by three countries, Kuwait, Libya and Venezuela; the available quantities are 24, 6 and 11 thousand ton per month, respectively. Variable costs (inclusive transportation costs) per unit supplied by a given country to a given refinery are as follows:

	Rotterdam	Southampton	Yokohama
Kuwait	20	18	25
Libya	12	11	33
Venezuela	17	17	40

- Construct the mathematical model of this problem.
- What is the optimal supply schedule for the refineries?

Exercise 2:

A company has plants in A, B and C and warehouses in I, II, III and IV. Of a certain product, 5, 10 and 12 units are available in A, B and C, respectively, and 8, 7, 5 and 7 units are required in I, II, III and IV. Transport costs per unit are given by the following table:

	I	II	III	IV
A	9	3	14	4
B	17	19	6	19
C	8	10	24	9

- Formulate the problem.
- Use the simplex transportation method to determine the supply program with minimum cost.

3. The Assignment Problem

Let us consider a company with a number of jobs to be filled and the same number of people who should fill these jobs. For each person it is known how well he does on each job by means of a performance coefficient which may be the monetary gain for the company. For instance, let there be four jobs and four persons and let the coefficients of the gain of the various persons filling the jobs be given by the following table:

		Jobs			
		1	2	3	4
Persons	1	4	2	0	3
	2	2	4	4	3
	3	2	6	2	3
	4	4	5	3	4

Hence, if person 1 fills job 1, the gain is 4, if he fills job 2, the gain is 2, if he fills job 3, the gain is 0, and so on.

Now, let us indicate person 1 filling job 1 as $x_{11} = 1$ and person 1 not filling job 1 as $x_{11} = 0$, person 1 filling job 2 as $x_{12} = 1$ and the same person not filling job 2 as $x_{12} = 0$. We assume that it is not possible that a person partly fills one job and part / another. The problem may then be formulated as follows. Maximize

$$f = 4x_{11} + 2x_{12} + 0x_{13} + 3x_{14} + \dots + 4x_{44} \quad (1)$$

Subject to

$$\begin{aligned} \sum_{j=1}^4 x_{1j} &= 1 \\ \sum_{j=1}^4 x_{2j} &= 1 \\ \sum_{j=1}^4 x_{3j} &= 1 \\ \sum_{j=1}^4 x_{4j} &= 1 \end{aligned}$$

(2)

$$\begin{aligned}
 \sum_{i=1}^4 x_{i1} &= 1 \quad . \\
 \sum_{i=1}^4 x_{i2} &= 1 \quad . \\
 \sum_{i=1}^4 x_{i3} &= 1 \quad . \\
 \sum_{i=1}^4 x_{i4} &= 1 \quad .
 \end{aligned}
 \quad \left. \vphantom{\sum_{i=1}^4} \right\} (3)$$

$$x_{ij} \geq 0 \quad (i = 1, \dots, 4 ; j = 1, \dots, 4) \quad (4)$$

$$x_{ij} = 0 \text{ or } 1 \quad (i = 1, \dots, 4 ; j = 1, \dots, 4) \quad (5)$$

The constraints (2) denote that each person can take only one job and those of (3) denote that the same job can be done only once. The constraints (4) and (5) reflect that no job can be performed at a negative level and that each person has to restrict himself to one job.

Except the constraints (5), the problem is an integer programming problem and the fact that this is a maximization problem instead of a minimization problem this is a transportation problem of a somewhat special structure, since what may be interpreted as required and available quantities are all 1. However, the difference between a maximization problem and a minimization problem can be transformed from one into the other by a multiplication by -1. Furthermore, it can be proved that the solution of a transportation problem with available and required

quantities that are integers, has always integer values. This means that any solution of the problem (1) to (4) without (5) will always satisfy (5). Hence the method of solving the transportation problem can be applied also here to the assignment problem.

Chapter 2

Integer Programming

A. Introduction

If we consider the L.P. problem

$$\max z = \sum c x$$

$$\text{s.t.} \quad Ax \leq b.$$

$$x \geq 0.$$

Or, by adding the slack variables $x_{n+1}, \dots, x_{n+m}$, we obtain the equations form:

$$\max x_0 = a_{00} - a_{01} x_1 - a_{02} x_2 - \dots - a_{0n} x_n$$

$$\text{s.t.} \quad x_{n+1} = a_{n+1,0} - a_{n+1,1} x_1 - a_{n+1,2} x_2 - \dots - a_{n+1,n} x_n$$

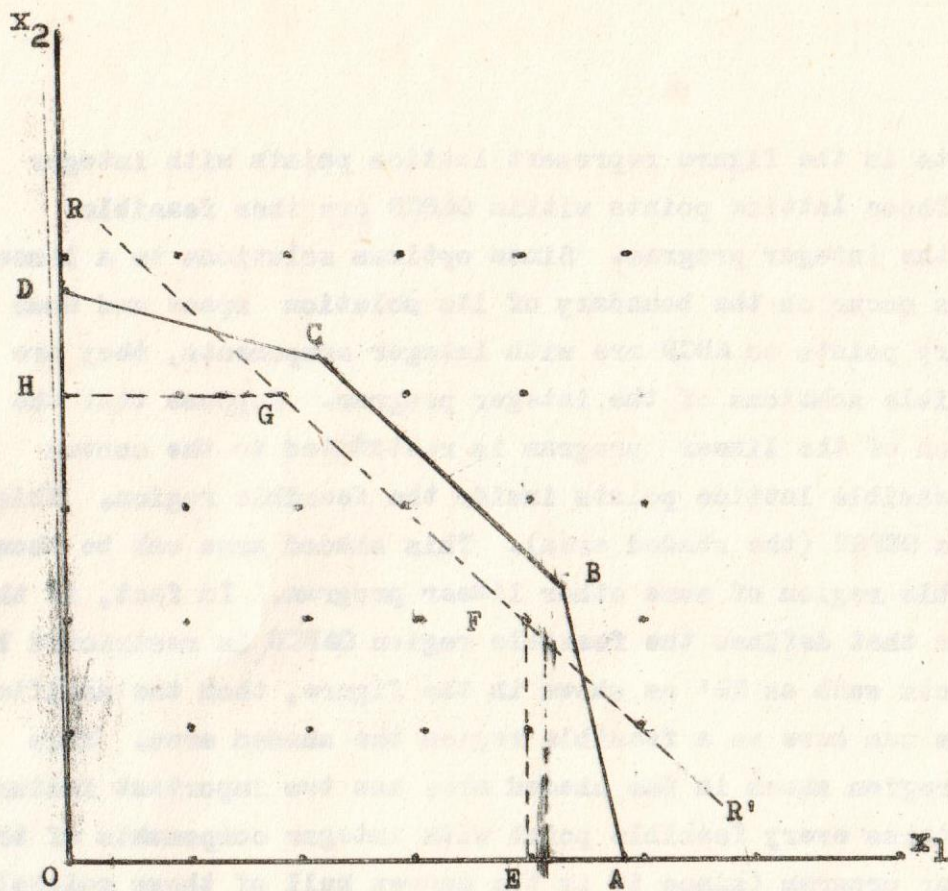
$$x_{n+m} = a_{n+m,0} - a_{n+m,1} x_1 - a_{n+m,2} x_2 - \dots - a_{n+m,n} x_n$$

(1)

$$x_j \geq 0 \quad (j = 1, \dots, n, n+1, \dots, n+m).$$

If, in addition to the constraints in (1), we require that all the original variables x_j ($j = 1, \dots, n$) must be integers, then the problem is called an integer linear programming problem (or briefly, an integer program). This problem is suggested by R.E. Gomory¹⁾.

1) Gomory, R.E., An Algorithm for Integer Solutions to Linear Programs, Princeton IBM Math. Res. Report, Nov. 1958.
also in R.L. Graves and P. Wolfe (eds.), Recent advances in Mathematical Programming, McGraw-Hill, N.Y., 1963.



The dots in the figure represent lattice points with integer components. Those lattice points within OABCD are then feasible solutions to the integer program. Since optimum solutions to a linear program always occur on the boundary of its solution space and none of the boundary points on ABCD are with integer components, they are not feasible solutions of the integer program. Suppose that the feasible region of the linear program is restricted to the convex hull of its feasible lattice points inside the feasible region. This convex hull is OEFGH (the shaded area). This shaded area can be thought of as a feasible region of some other linear program. In fact, if the linear program that defines the feasible region OABCD is restricted by more constraints such as RR' as shown in the figure, then the modified linear program can have as a feasible region the shaded area. This new feasible region shown in the shaded area has two important features: First, it contains every feasible point with integer components of the original linear program (since it is the convex hull of those points). Second, all the extreme points of this new feasible region have integer components. Therefore any basic optimum solution of the modified linear program has integer components and it is also the optimum solution to the original integer program.

Thus the integer programming algorithms are just a methods of systematically generating additional constraints to cut the feasible region down to the convex hull of its integer feasible points.

As soon as this is done, we can solve the modified linear program by any known method, and its basic optimum solution will automatically be an integer solution. The integer algorithm presented here will have the following properties:

1. Generated constraints never exclude from the new feasible region an integer feasible point of the original integer program.
2. In a finite number of steps, enough constraints will be generated to cut the feasible region of the original linear program so that the optimum solution of the modified linear program has integer components.

3. The generated constraints (or hyperplanes) pass through at least one integer point, although not necessarily an integer point inside the convex hull.
4. Each new constraint will reduce the feasible region of the original linear program. At this point, it should be considered that before the feasible region of the original linear program is cut to its convex hull, we may already obtain the optimum solution to the original integer program. Also, since the optimum integer solution can be defined as the intersection of n hyperplanes, these n hyperplanes are actually the only ones that are really needed; some of these may be constraints of the original problem.

Now, we introduce n trivial equations $x_j = -(-x_j)$ ($j = 1, \dots, n$) as part of the constraints of (1) and then write the integer program in a matrix form as

$$\underline{x} = \underline{A} (-\underline{x}_n) \quad (2)$$

where $\underline{x}$ is a column vector with $x_0, x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m}$ as

Components,

$\underline{A}$ is a matrix of $(m + m + 1) \times (n + 1)$, and $(-\underline{x}_n)$ is the vector with its components $(1, -x_1, -x_2, \dots, -x_n)$, where $x_1, x_2, \dots, x_n$ are the nonbasic variables in the starting tableau.

The integer program also can be put in the form of the following tableau:

	1	$-x_1$	...	$-x_n$
$x_0 =$	a_{00}	a_{01}	...	a_{0n}
$x_1 =$	0	-1		
$\vdots$	$\vdots$			
$x_n =$	0			-1
$x_{n+1} =$	$a_{n+1,0}$	$a_{n+1,1}$	...	$a_{n+1,n}$
$\vdots$	$\vdots$	$\vdots$		
$x_{n+m} =$	$a_{n+m,0}$	$a_{n+m,1}$	...	$a_{n+m,n}$

We shall use a_{ij} ($j = 0, 1, \dots, n$) to denote the column vector with all its components in the j th column of the current tableau and a_{ij} ($i = 0, 1, \dots, n+m; j = 0, 1, \dots, n$) the coefficients in the i th row and j th column of the tableau. We shall assume that all a_{ij} are integers in the starting tableau. This implies that the slack variables $x_{n+1}, \dots, x_{n+m}$ should also be nonnegative integers.

R.E. Gomory (in "An Algorithm for Integer Solutions to Linear Programs," Princeton IBM Math. Res. Report, Nov. 1958; and later on in R.L. Graves and P. Wolfe (eds.), Recent Advances in Mathematical Programming, McGraw Hill, New York, 1963) described the following algorithm for solving integer programs.

2. Gomory's Algorithm

This algorithm first treats the integer program as a linear program and solves it by the simplex method. At the end of the computation, we have $a_{i0} \geq 0$ ($i = 1, \dots, n+m$) and $a_{j0} \geq 0$ ($j = 1, \dots, n$).

If $a_{i0} > 0$ and are all integers, then this is obviously also the optimum solution to the integer program, as the integer constraints were neglected in obtaining the solution. If $a_{i0} \geq 0$ but not all integers, we add a constraint in addition to (1).

We shall begin our method by considering the pure integer programming problem (ILP):

$$\text{Maximize } Z = c'x \quad (3)_a$$

$$\text{s.t. } Ax = b, \quad (3)_b$$

$$x \geq 0,$$

$$x_j \text{ integer, } j = 1, 2, \dots, n \quad (3)_c$$

The problem with the integer constraints $(3)_c$ deleted shall be called the related linear programming problem. The basic idea in the algorithm is the following: Given an integer programming problem, solve the related LP problem. If the optimal solution contains only integer-valued variables, clearly, it must be the optimal ILP solution. If

some variables in the optimal L P solution are not integers, a new constraint is added to the problem, and the new related L P problem is solved. Again, we check to see if the optimal solution to this new L P problem satisfies the integer requirements; if so, it is the optimal solution for the original I L P problem. If, on the other hand, some of the variables are again not integers, then another new constraint is added, and the process is repeated until one of the related L P problems is found to have an all integer optimal solution. The first such all integer solution obtained is the optimal solution for the original problem.

In order for the above process to be valid, the new constraint which is added at each step must possess the following properties:

- i) Every feasible integer solution to the original problem must also be a feasible solution to the new problem resulting from the addition of the constraint, i.e., the addition of the new constraint does not eliminate any of the integer solutions from the set of feasible solutions, thus ensuring that the optimal integer solution will not be made infeasible.
- ii) The optimal solution to the L P problem solved at each step must become infeasible after the new constraint is added. This property is necessary because if the addition of the constraint did not eliminate the optimal L P solution, the latter would also be optimal for the new L P problem. Since we already know the former optimal L P solution is not an all-integer solution, nothing will have been gained by adding a constraint in which this solution is not excluded.

Each of the constraints added in the above manner are called cutting plane constraints, because they are hyperplanes which cut off a portion of the convex set of feasible solutions for the related L P problem. Therefore, this method is called, Gomory's cutting plane method.

Now, let us investigate the nature of the particular cutting plane constraints devised by Gomory for solving the pure integer programming problem (3). For simplicity, assume that the optimal solution to the related LP problem contains the first m columns of A in the basis. Hence, A can be partitioned into two submatrices $[B, R]$, where B contains the first m columns of A (the basis matrix) and R contains the remaining $(n - m)$ columns of A . The corresponding optimal solution is $x^* = [x_B^*, x_R^*]$, where $x_R^* = 0$. Now, since $Ax = b$, we have

$$[B, R] \begin{bmatrix} x_B^* \\ x_R^* \end{bmatrix} = b \quad (4)$$

$$B x_B^* + R x_R^* = b$$

Premultiplying equation (4) by B^{-1} gives

$$\begin{aligned} x_B^* + (B^{-1}R) x_R^* &= B^{-1}b \\ \text{or} \quad x_B^* &= B^{-1}b - (B^{-1}R) x_R^* \end{aligned} \quad (5)$$

Since R consists of the columns of A , we can express equation (5) in terms of the columns of the simplex tableau; namely, letting

$$y_j = B^{-1}a_j \quad \text{and} \quad y_0 = B^{-1}b,$$

we have

$$x_B^* = y_0 - \sum_{j \in N} y_j x_j \quad (6)$$

where

$$\begin{aligned} N &= \{ j / a_j \text{ is a column of } R \} \\ &= \{ j / x_j \text{ is nonbasic} \} \end{aligned}$$

Suppose now that the r th basic variable of x_B^* is not an integer. From (6), we see that

$$x_{Br} = y_{r0} - \sum_{j \in N} y_{rj} x_j \quad (7)$$

In the current solution, all x_j , $j \in N$, are zero, and so y_{r_0} is not an integer. Let us express y_{r_0} as the sum of its integer part and its fractional part. Let w_{r_0} be the integer (or whole) part of y_{r_0} (i.e., the greatest integer less than or equal to y_{r_0}) and let f_{r_0} be the remaining fractional part. Thus,

$$y_{r_0} = w_{r_0} + f_{r_0} \quad (8)_a$$

$$w_{r_0} \geq 0 \quad (8)_b$$

$$0 < f_{r_0} < 1 \quad (8)_c$$

Similarly, we represent the y_{r_j} , $j \in N$, as:

$$y_{r_j} = w_{r_j} + f_{r_j} \quad (9)_a$$

$$0 \leq f_{r_j} < 1 \quad (9)_b$$

If we substitute equations $(8)_a$ and $(9)_a$ into equation (7), we obtain

$$\begin{aligned} x_{Br} &= w_{r_0} + f_{r_0} - \sum_{j \in N} (w_{r_j} + f_{r_j}) x_j \\ &= (w_{r_0} - \sum_{j \in N} w_{r_j} x_j) + (f_{r_0} - \sum_{j \in N} f_{r_j} x_j) \quad (10)_a \end{aligned}$$

Now, if we wish to modify the current solution so that the new solution is all-integer, then at least one of the currently nonbasic x_j ($j \in N$) must become positive (since there is a unique solution with all these $x_j = 0$, and it is x_B^v). In such an all integer solution, the first term in equation $(10)_a$ will be an integer, since each x_j must be an integer, and each w_{r_j} is an integer by definition. Thus, the new value of x_{Br} can be expressed by

$$x_{Br} = \text{integer} + (f_{r_0} - \sum_{j \in N} f_{r_j} x_j) \quad (10)_b$$

Moreover, we also desire x_{Br} to be an integer; if this is to be attained, then the quantity $(f_{r_0} - \sum_{j \in N} f_{r_j} x_j)$ must be an integer. Since each $f_{r_j} \geq 0$ and each $x_j \geq 0$, it is clear that

$$\sum_{j \in N} f_{r_j} x_j \geq 0$$

Hence, the quantity $(f_{r_0} - \sum_{j \in N} f_{r_j} x_j)$ consists of one positive quantity, $\sum_{j \in N} f_{r_j} x_j$, subtracted from a positive quantity f_{r_0} , which is less than one (by definition, see equation (8)_c).

The entire quantity $(f_{r_0} - \sum_{j \in N} f_{r_j} x_j)$, therefore, cannot be a positive integer. If it is to be an integer, then, it must be a negative integer (or zero):

$$f_{r_0} - \sum_{j \in N} f_{r_j} x_j \leq 0 \quad (11)$$

The inequality (11) is the cutting plane constraint derived by Gomory. Noticing that, it satisfies the two properties (i) and (ii).

Up to now, we have not considered how to determine the cutting plane constraint if more than one basis variable is not an integer. In the familiar technique the x_{Br} whose fractional part f_{r_0} is the largest is used to obtain the constraint (11). However, there are many other possible ways to select the cutting plane constraint.

Gomory proved that with the proper choice for the cutting plane constraint at each step, the algorithm converges¹⁾ to the optimal

1) Hadley, G.; Nonlinear and Dynamic Programming. Reading, Mass., Addison-Wesley, 1964.

integer solution in a finite number of iterations. Hadley mentioned also that it is not necessary to have more than $(n+1)$ constraints at one time in the related L P problem; that is, under certain conditions, it is possible to drop constraints previously added, so that the total number of constraints never exceeds $(n+1)$.

When we add the constraint (11) to the optimal tableau for the related L P problem, the resulting problem is solved by the simplex method.

Example:

Consider the following problem

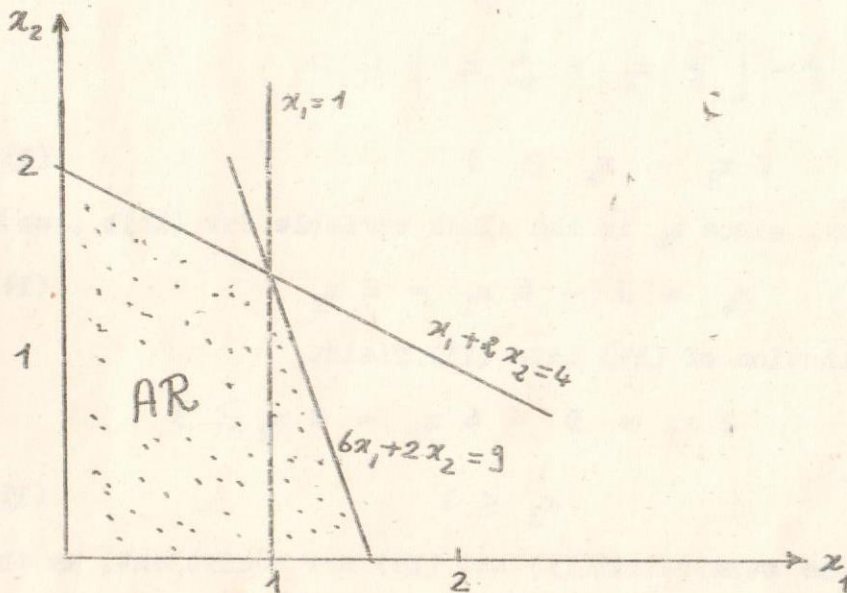
$$\text{Maximize } Z = x_1 - x_2$$

$$\text{s.t. } x_1 + 2x_2 \leq 4 \quad (12)_a$$

$$6x_1 + 2x_2 \leq 9 \quad (12)_b$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \text{ integer}$$



The optimal solution to the related L P problem occurs at $(x_1, x_2) = (\frac{3}{2}, 0)$. The optimal simplex tableau is the following table (where x_3 and x_4 are the slack variables added to the constraints (12)_a and (12)_b respectively as usual):

Basis	b	x_1	x_2	x_3	x_4
$x_{B1} = x_1$	3/2	1	2/6	0	1/6
$x_{B2} = x_3$	5/2	0	10/6	1	-1/6
Z	3/2	0	8/6	0	1/6

Since, both x_{B1} and x_{B2} are noninteger, we may use either in the calculation of cutting plane (11). Arbitrarily, we choose x_{B1} .

From the table, we see that $y_{12} = \frac{2}{6}$.

$y_{14} = \frac{1}{6}$ and $y_{10} = \frac{3}{2} = 1 + \frac{1}{2}$. Thus (11) becomes

$$\frac{1}{2} - \left[\frac{2}{6} x_2 + \frac{1}{6} x_4 \right] \leq 0$$

or

$$2 x_2 + x_4 \geq 3 \quad (13)$$

However, since x_4 is the slack variable for (12)_b, we have

$$x_4 = 9 - 6 x_1 - 2 x_2 \quad (14)$$

Substitution of (14) into (13) yields

$$2 x_2 + 9 - 6 x_1 - 2 x_2 \geq 3$$

i.e.

$$x_1 \leq 1 \quad (15)$$

Now, the constraints (13) and (15) are equivalent, so that we can graph this constraint, (15), in terms of x_1 and x_2 . the cutting plane constraint (15) is shown in the figure, and the optimal solution to this new related L P Problem occurs at $(x_1, x_2) = (1, 0)$, which is the required integer solution.

However, suppose instead that we had chosen to calculate the cutting plane constraint based on x_{B2} . From the table, we see that $y_{22} = \frac{10}{6}$, $y_{24} = -\frac{1}{6}$, and $y_{20} = \frac{5}{2}$. Hence $f_{22} = \frac{4}{6}$, $f_2 = \frac{5}{6}$ (since f_{B1} must be non-negative, and $y_{24} = -\frac{1}{6} = -1 + \frac{5}{6}$). The new constraint would then have been

$$\frac{1}{2} - \left[\frac{4}{6} x_2 + \frac{5}{6} x_4 \right] \geq 0$$

or

$$4 x_2 + 5 x_4 \geq 3 \quad (16)$$

Again, we eliminate x_4 by substituting equation (14) into (16), yielding

$$4 x_2 + 5 (9 - 6 x_1 - 2 x_2) \geq 3$$

or

$$5 x_1 + x_2 \leq 7 \quad (17)$$

If (17) is added to the original problem, instead of (15), the resulting LP problem has an optimal solution of $(x_1, x_2) = (\frac{7}{5}, 0)$, which is not all integer. Thus, the choice of constraints to be added affects the efficiency of the algorithm. Unfortunately, there is no apriori method of determining the best choice of a cutting plane constraint.

3. A Branch and Bound Algorithm

The final topic we wish to discuss in this chapter is branch and bound methods. In this section, we shall study Dakin's method,¹⁾ which is a modification of the original branch and bound algorithm proposed by Land and Doig²⁾. This method is the most effective technique for solving integer programming problems, moreover, it is the basic

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- 1) Dakin, R.J.; A tree - search algorithm for mixed integer programming problems. Computer Journal, 9, 1966.
 - 2) Land, A.H., and Doig, A.; An automatic method of solving discrete programming problems. Econometrica, 28, 1960.

approach of computer programs for solving integer programming problems. The method can be used for either pure or mixed integer programming problems. In general, suppose we have to solve the problem

$$\begin{array}{ll}\text{Maximize} & Z = c'x \\ \text{s.t.} & Ax = b, \\ & x \geq 0, \\ & x_j \text{ integer, } j \in I\end{array}$$

where I is the set of subscripts for those variables required to be integer; if $I = \{1, 2, \dots, n\}$, the problem is a pure integer programming problem. The outline of the algorithm is as follows:

1. Solve the related LP problem. If all x_j , $j \in I$, are integers, we have obtained the desired optimal solution. If at least one x_j , $j \in I$, is non-integer, go to step 2.
2. Select one of the non-integer x_j , $j \in I$. Suppose

$$x_j = w_j + f_j \quad (1)$$

where w_j is an integer and

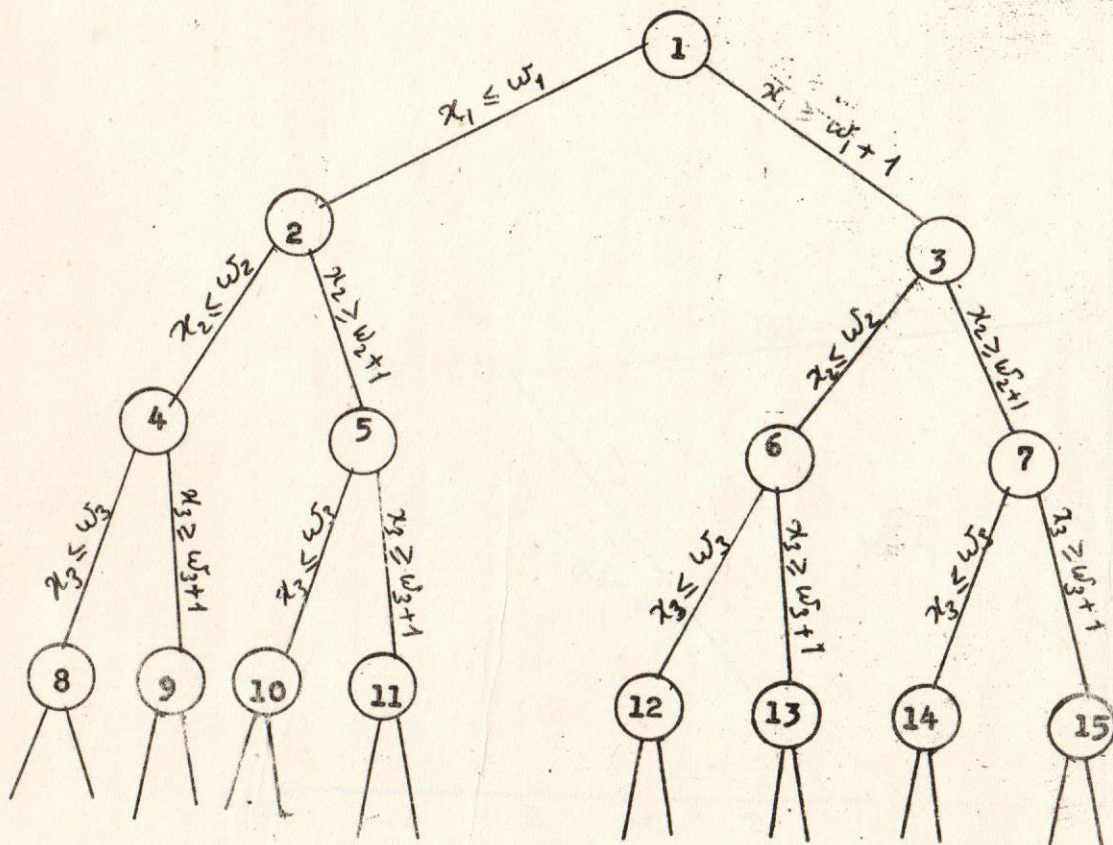
$$0 < f_j < 1 \quad (2)$$

Any feasible integer solution must therefore satisfy either $x_j \leq w_j$ or $x_j \geq w_j + 1$. We add each of these constraints, individually, to the constraint set, and solve the resulting two related LP problems.

3. For each of the two solutions found in step 2:
 - a) If the solution is a feasible integer solution, compare its objective function value with the best feasible integer solution found thus far. Save the better of the two. Select another non-integer x_j , $j \in I$, if any exists, and return to Step 2.

- b) If the resulting L P problem has no feasible solution, select another non - integer x_j , $j \in I$, if any exists, and return to Step 2.
- c) If the solution is not a feasible integer solution, repeat the process of Step 2 on the constraint set for this related L P problem.
4. The algorithm terminates when all possibilities have been exhausted in Step 3.

The algorithm generates a tree of solutions as shown in the following figure



Before going further describing the algorithm in more detail, we prefer to illustrate it with an example.

Example:

Consider the following problem

$$\text{Maximize } Z = x_1 + x_2$$

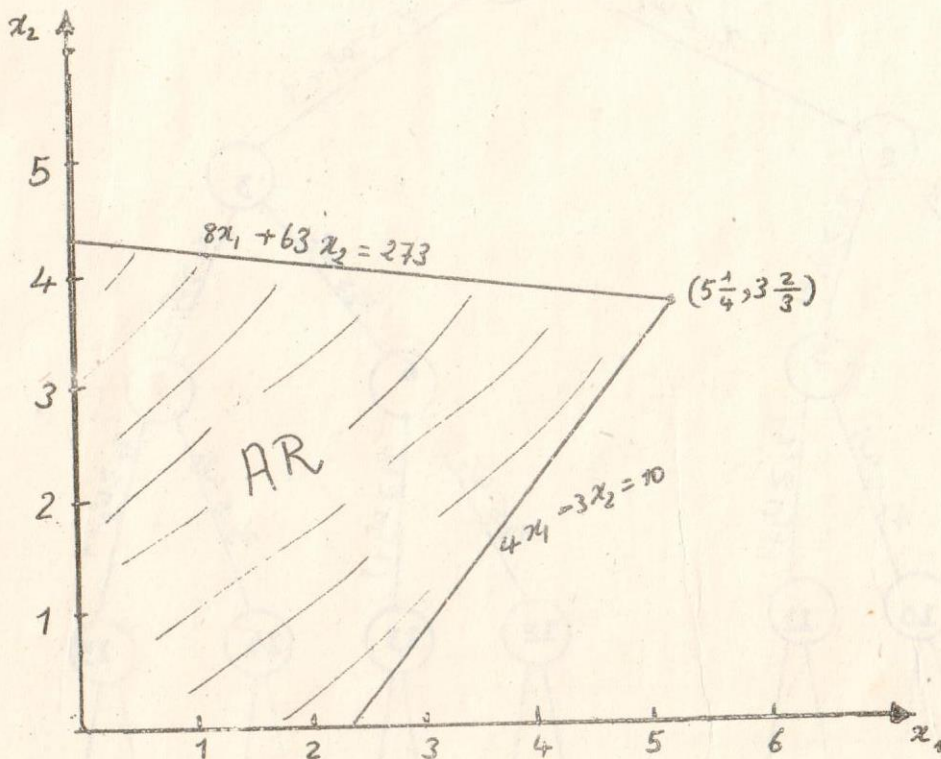
s.t.

$$8x_1 + 63x_2 \leq 273$$

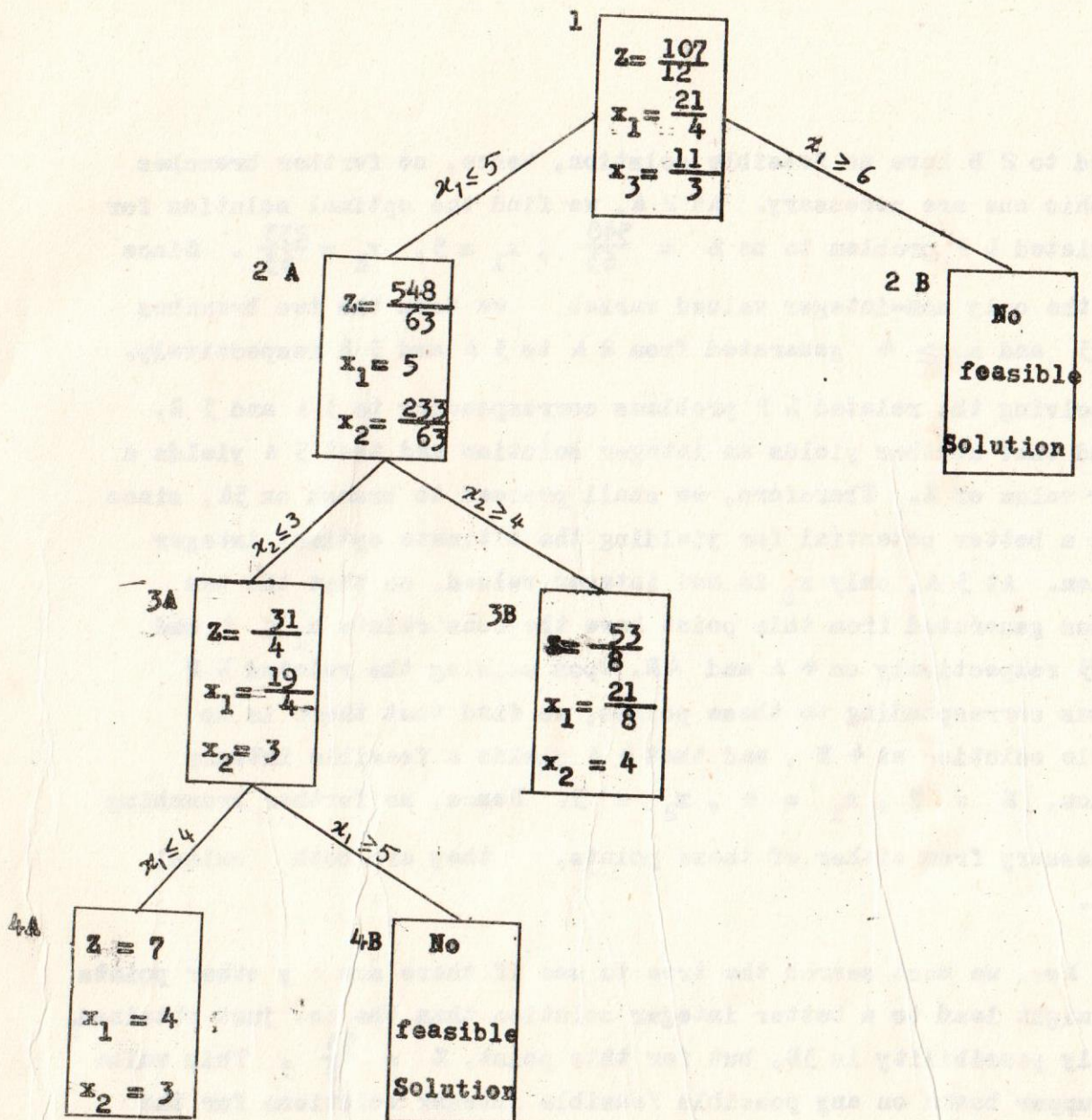
$$4x_1 - 3x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \text{ integer}$$



The tree of solutions generated by the procedure below is given in the following figure:

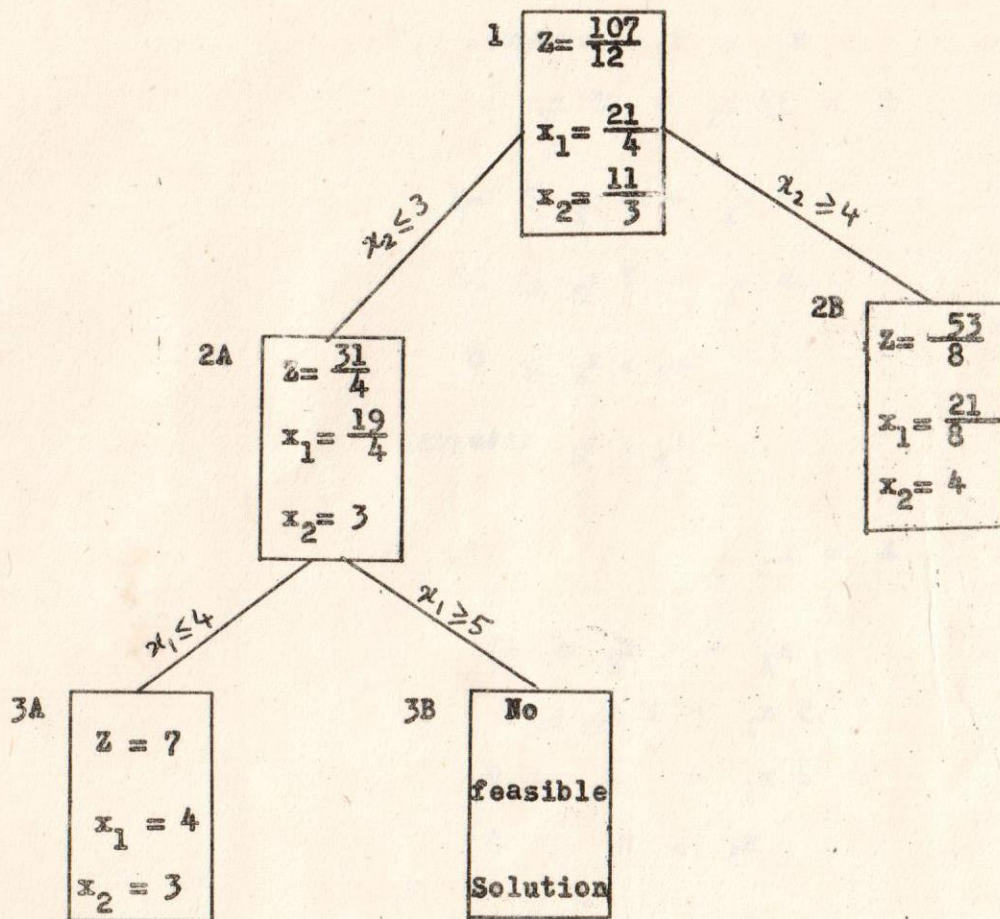


We begin by solving the related L P problem. (Step 1 of our outline) which yields $Z = \frac{107}{12}$, $x_1 = \frac{21}{4}$, $x_2 = \frac{11}{3}$ (labeled 1 in the tree of solutions). Since neither x_1 nor x_2 is integer, we can choose either variable to begin implementation of Step 2. Arbitrarily, we select x_1 , yielding the two branches $x_1 \leq 5$ and $x_1 \geq 6$, to 2 A and 2B respectively. The constraints thus

imposed to 2 B have no feasible solution, hence, no further branches from this one are necessary. At 2 A, we find the optimal solution for the related L P problem to be $Z = \frac{548}{63}$, $x_1 = 5$, $x_2 = \frac{233}{63}$. Since x_2 is the only non-integer valued variable, we form the two branches $x_2 \leq 3$ and $x_2 \geq 4$ generated from 2 A to 3 A and 3 B respectively. Upon solving the related L P problems corresponding to 3 A and 3 B, we find that neither yields an integer solution and that 3 A yields a larger value of Z. Therefore, we shall proceed to branch on 3A, since it has a better potential for yielding the ultimate optimal integer solution. At 3 A, only x_1 is not integer valued, so that the two branches generated from this point have the constraints $x_1 \leq 4$ and $x_1 \geq 5$ respectively on 4 A and 4B. Upon solving the related L P problems corresponding to these points, we find that there is no feasible solution at 4 B, and that 4 A yields a feasible integer solution, $Z = 7$, $x_1 = 4$, $x_2 = 3$. Hence, no further branching is necessary from either of these points, as they are both terminal points.

Now, we must search the tree to see if there are any other points which might lead to a better integer solution than the one just obtained. The only possibility is 3B, but for this point, $Z = \frac{53}{8}$. This value is an upper bound on any possible feasible integer solutions for the constraints imposed at this point, and since we have already found a better integer solution, we now know that it must be the optimal integer solution. Hence, the algorithm terminates.

In the second step of the above procedure, we arbitrarily chose to branch on x_1 (i.e., the two branch constraints were $x_1 \leq 5$ and $x_1 \geq 6$). If, instead, we had branched on x_2 , we would have generated the tree of solutions given in the following figure:



Problems:

1) Solve the following problems by the cutting plane method, the branch and bound method and then illustrate it graphically.

a) Maximise $z = x_2 - x_1$
 s.t.
 $2x_1 + x_2 \leq 9,$
 $2x_1 + 6x_2 \leq 21,$
 $x_1, x_2 \geq 0.$

$$x_1, x_2 \text{ integer.}$$

b) Maximize $Z = 32 x_1 + 15 x_2$

s.t. $2 x_1 - 3 x_2 \geq -9$

$$16 x_1 + 7 x_2 \leq 52$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \text{ integer.}$$

c) Maximize $Z = x_1$

s.t. $x_1 + 2 x_2 \leq 12$

$$5 x_1 + 2 x_2 \leq 30$$

$$2 x_1 - x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \text{ integer.}$$

2) Solve by a branch and bound method:

Maximize $Z = x_1 + x_2$

s.t. $x_1 + 2 x_2 \leq 12$

$$5 x_1 + 2 x_2 \leq 30$$

$$2 x_1 - x_2 \leq 9$$

$$x_1, x_2 \geq 0$$

$$x_1, x_2 \text{ integer.}$$

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Chapter 3

Linear Programming Models

(1) Prove the convexity of the following functions:

a) $z = e^{x+y}$

b) $y = x_1^2 + 2x_1 x_2 + x_2^2 + 2x_1 - 3x_2 + 8.$

c) $y = 3x_1^2 + 4x_1 x_2 - 2x_2^2 + 8x_2.$

(2) Consider the following problem:

(P) $\min \quad \left\{ x_1^2 + x_2^2 \quad / \quad x_1 + x_2 \geq 1, \quad x_1 \leq 1, \right.$

$\left. 3x_1^2 + 4x_2^2 \leq 36, \quad x_1 \geq 0, \quad x_2 \geq 0 \right\}$

a) Is P a Convex Problem?

b) Get a geometrical interpretation of the problem P and find a solution?

c) Apply local kuhn - Tucker conditions?

Solution:

$z = e^{x+y}$

Convex?

Let $\lambda = x+y$

$\varphi(\lambda) = e^\lambda$

$\varphi(\alpha x_1 + (1-\alpha) x_2) \leq \alpha \varphi(x_1) + (1-\alpha) \varphi(x_2)$

Let $\alpha = 1/2$.

$$\varphi\left(\frac{x_1}{2} + \frac{x_2}{2}\right) \leq \frac{1}{2} (\varphi(x_1) + \varphi(x_2))$$

$$\varphi\left(\frac{\lambda_1}{2} + \frac{\lambda_2}{2}\right) \leq \frac{1}{2} (\varphi(\lambda_1) + \varphi(\lambda_2))$$

$$e^{\frac{1}{2}(\lambda_1 + \lambda_2)} \leq \frac{1}{2} (e^{\lambda_1} + e^{\lambda_2})$$

$$\left[1 + \frac{\lambda_1 + \lambda_2}{2} + \frac{(\lambda_1 + \lambda_2)^2}{2! \cdot 2!} + \dots \right] \leq \frac{1}{2} \left[1 + \frac{\lambda_1}{1} + \frac{\lambda_1^2}{2!} + \dots + 1 + \frac{\lambda_2}{1} + \frac{\lambda_2^2}{2!} + \dots \right]$$

$$\frac{1}{2} \frac{[\lambda_1 + \lambda_2]^2}{2!} \leq \frac{1}{2} \frac{[\lambda_1^2 + \lambda_2^2]}{2!}$$

$$[\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2] \leq 2[\lambda_1^2 + \lambda_2^2]$$

$$2\lambda_1\lambda_2 \leq \lambda_1^2 + \lambda_2^2$$

$$[\lambda_1 - \lambda_2]^2 \geq 0$$

i.e $Z = e^{x+y}$ is Convex

b) $y = x_1^2 + 2x_1x_2 + x_2^2 + 2x_1 - 3x_2 + 8$ Let $\varphi = (x+y)^2 + 2x - 3y + 8$

since the condition of convexity is:

$$\varphi(\alpha x_1 + (1-\alpha)x_2) \leq \alpha \varphi(x_1) + (1-\alpha)\varphi(x_2)$$

then:

$$\begin{aligned} 2 \left[\left(\frac{x_1 + x_2}{2} + \frac{y_1 + y_2}{2} \right)^2 + \frac{(x_1 + x_2)}{2} - \frac{3(y_1 + y_2)}{2} + 8 \right] &\leq (x_1 + y_1)^2 \\ &+ 2x_1 - 3y_1 + 8 + (x_2 + y_2)^2 + 2x_2 - 3y_2 + 8 \end{aligned}$$

$$(x_1 + y_1 + x_2 + y_2)^2 \leq 2(x_1 + y_1)^2 + 2(x_2 + y_2)^2$$

$$(x_1 + y_1)^2 + 2(x_1 + y_1)(x_2 + y_2) + (x_2 + y_2)^2 \leq 2(x_1 + y_1)^2 + 2(x_2 + y_2)^2$$

$$[(x_1 + y_1) - (x_2 + y_2)]^2 \geq 0$$

i.e. it is convex.

$$c) \quad y = 3x_1^2 + 4x_1 x_2 - 2x_2^2 + 8x_2$$

$$\left\{ y = x^T C x + d^T x \right\}$$

$$\frac{\partial^2 \varphi(x)}{\partial x_i \partial x_k} = A \begin{cases} \text{positive semidefinite} \Rightarrow \varphi(x) \text{ (Conv.)} \\ \text{positive definite} \Rightarrow \varphi(x) \text{ (cont. Conv.)} \end{cases}$$

$$C = \begin{pmatrix} 3 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\frac{\partial \varphi}{\partial x_1} = 6x_1 + 4x_2 \quad ; \quad \frac{\partial^2 \varphi}{\partial x_1 \partial x_1} = 6, \quad \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = 4$$

$$\Rightarrow 2C$$

$$\frac{\partial \varphi}{\partial x_2} = 4x_1 - 4x_2 + 8; \quad \frac{\partial^2 \varphi}{\partial x_2 \partial x_1} = 4, \quad \frac{\partial^2 \varphi}{\partial x_1 \partial x_2} = 4$$

$$[A] = -40 \Rightarrow A \text{ is not p.s.d., i.e., } y \text{ is not convex.}$$

$$(2) \quad \min \left\{ x_1^2 + x_2^2 / x_1 + x_2 \geq 1, \quad x_1 \leq 1, \quad 3x_1^2 + 4x_2^2 \leq 36, \right.$$

$$\left. x_1 \geq 0, \quad x_2 \geq 0 \right\}$$

a) to prove the convexity?

$$F(x) = x_1^2 + x_2^2 = \min !$$

$$\frac{\partial F}{\partial x_1} = 2x_1, \quad \frac{\partial^2 F}{\partial x_1 \partial x_1} = 2, \quad \frac{\partial^2 F}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial F}{\partial x_2} = 2x_2, \quad \frac{\partial^2 F}{\partial x_2 \partial x_1} = 0, \quad \frac{\partial^2 F}{\partial x_2 \partial x_2} = 2$$

i.e. $[A]$ positive definite

then $F(x)$ is Convex (the objective function).

Now, let

$$f_1 = 1 - x_1 - x_2 \leq 0$$

$$\frac{\partial f_1}{\partial x_1} = -1, \quad \frac{\partial f_1}{\partial x_2} = -1$$

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \text{ is a positive semidefinite}$$

then $f_1(x)$ is convex.

$$\text{let } f_2 = x_1 - x_2 \leq 0$$

$$\frac{\partial f_2}{\partial x_1} = 1, \quad \frac{\partial f_2}{\partial x_2} = -1$$

$$\begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} \text{ positive semidefinite, i.e.}$$

$f_2(x)$ is convex.

let

$$f_3 = 3x_1^2 + 4x_2^2 - 36 \leq 0$$

$$\frac{\partial f_2}{\partial x_1} = 6x_1$$

$$\frac{\partial f_2}{\partial x_2} = x_2$$

$$\begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} \Rightarrow \text{def. } f_2(x) \text{ is Convex.}$$

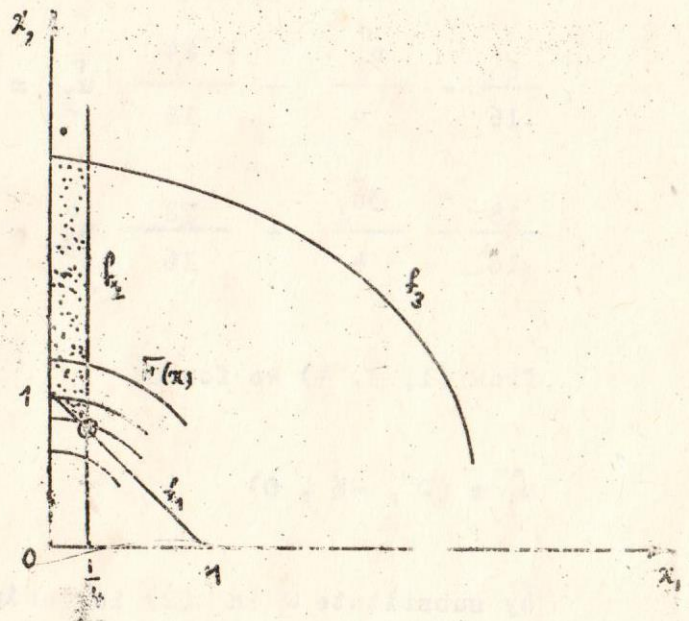
∴ P is a convex problem.

b) gem. Interpr.

From the Fig. we find by trial and error the solution:

$$x_0 = (1/4, 1/4)$$

$$\text{which gives } F(x) = \frac{5}{8}$$



c) Local Kuhn-Tucker Conditions:

$$\Phi(x^0, u^0) \geq 0$$

$$x^{0'} \Phi_x(x^0, u^0) = 0$$

$$\Phi_u(x^0, u^0) \leq 0, u^{0'} \Phi_u(x^0, u^0) = 0; x^0 \geq 0, u^0 \geq 0.$$

$$\Phi(x) = F(x) + u_1 f_1(x) + u_2 f_2(x) + u_3 f_3(x)$$

$$= x_1^2 + x_2^2 + u_1 (1 - x_1 - x_2) + u_2 (x_1 - 1/4) + u_3 (3x_1^2 + 4x_2^2 - 36)$$

$$\Phi_x = (2x_1 - u_1 + u_2 + 6u_3 x_1, 2x_2 + u_1 + 8u_3 x_2)$$

$$\Phi_u = ((1 - x_1 - x_2), (x_1 - 1/4), (3x_1^2 + 4x_2^2 - 36))$$

$$\therefore x^{0'} \Phi_x(x^0, u^0) = 0$$

$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{4} - u_1^o + u_2^o + \frac{6}{4} u_3^o & \frac{6}{4} - u_1^o + \frac{24}{4} u_3^o \end{bmatrix} = 0$$

or $\frac{2}{16} - \frac{u_1^o}{4} + \frac{u_2^o}{4} + \frac{6}{16} u_3^o = 0 \quad (1)$

$$\frac{6}{16} - \frac{3u_1^o}{4} + \frac{3u_2^o}{4} + \frac{18}{16} u_3^o = 0 \quad (2)$$

$$\frac{6}{16} - \frac{u_1^o}{4} + \frac{24}{16} u_3^o = 0 \quad (3)$$

$$\frac{18}{16} - \frac{3u_1^o}{4} + \frac{72}{16} u_3^o = 0 \quad (4)$$

from (1, 3, 4) we found:

$$u_1^o = (0, -\frac{1}{2}, 0)$$

by substitute u in (2) is verified.

$$\therefore \lambda^o \Phi_x(x^o, u^o) = 0 \quad " \quad "$$

$$\Phi_x(x^o, u^o) = (0, \frac{3}{2}) \geq 0$$

$$\Phi_u(x^o, u^o) = (0, 0, -\frac{537}{16}) \leq 0$$

$$u^o \Phi_u(x^o, u^o) = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{537}{16} \end{bmatrix} = 0$$

i.e. the Local Kuhn-Tucker conditions are satisfied.

(3) Solve the following linear programming problem

$$2x - y \leq 3$$

$$x + 2y \leq 4$$

$$-x + y \leq 0$$

$$\min \quad x - y = \mu$$

Or, the problem is converted to:

$$2x - y + z_1 = 3$$

$$x + 2y + z_2 = 4$$

$$-x + y + z_3 = 0$$

$$\min \quad x - y$$

		x	y	z_1	z_2	z_3		
		1	-1	0	0	0	✓	
0	z_1	2	-1	1	0	0	3	
0	z_2	1	2	0	1	0	4	2
0	z_3	-1	(+1)	0	0	1	0	0
	f_0	0	0	0	0	0		
	$z - f_0$	4	(-1)	0	0	0		
0	z_1	1	0	1	0	1	3	
0	z_2	3	0	0	1	-2	4	
-1	y	-1	1	0	0	1	0	
	z_1	+1	-1	0	0	-1	0	
	$z - f$	0	0	0	0	+1	0	

Then, we found the solution as follows:

$$\min \mu = 0$$

$$x = 0$$

$$y = 0$$

$$z_1 = 3$$

$$z_2 = 4$$

$$z_3 = 0$$

(4) Solve the following linear programming problem:

$$\text{Max } (x_1 - x_2 + 2x_4 + 2x_5 + 2x_6)$$

$$-x_1 + x_2 + x_3 - 2x_4 - 2x_6 \leq -2$$

$$x_1 - x_4 + x_5 - 4x_6 \leq -1$$

$$2x_2 + 2x_4 + x_6 \leq 2$$

$$3x_2 + x_3 - x_4 + x_5 - 5x_6 \leq -1$$

$$; x_i \geq 0, i=1, \dots, 6.$$

Solution:

The previous problem is equivalent to :

$$-x_1 + x_2 + x_3 - 2x_4$$

$$-2x_6 + y_1 = -2$$

$$x_1 - x_4 + x_5$$

$$-4x_6 + y_2 = -1$$

$$2x_2 + 2x_4$$

$$+ x_6 + y_3 = 2$$

$$3x_2 + x_3 - x_4 + x_5$$

$$-5x_6 + y_4 = -1$$

Or

$$-x_1 - x_2 - x_3 + 2x_4$$

$$+ 2x_6 - y_1 + z_1 = 2$$

$$-x_1 + x_4 - x_5$$

$$+ 4x_6 - y_2 + z_2 = 1$$

$$2x_2 + 2x_4 + x_6 + y_3 = 2$$

$$-3x_2 - x_3 + x_4 - x_5 + 5x_6 - y_4 + z_3 = 1$$

with

$$\varphi = z_1 + z_2 + z_3 = 4 + 4x_2 + 2x_3 - 4x_4 + 2x_5 - 11x_6 + y_1 + y_2 + y_4$$

Basis	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z_1	z_2	z_3	K
z_1	1	-1	-1	2	0	2	-1	0	0	0	1	0	0	2
z_2	-1	0	0	1	-1	4	0	-1	0	0	0	1	0	1
y_3	0	2	0	2	0	1	0	0	1	0	0	0	0	2
z_4	0	-3	-1	1	-1	(5)	0	0	0	-1	0	0	1	1
$-z$	1	-1	0	2	2	2	0	0	0	0	0	0	0	0
$-\varphi$	0	4	2	-4	2	-11	1	1	1	0	0	0	0	-4
z_1	1	1/5	3/5	8/5	2/5	0	-1	0	0	7/5	1	0	-2/5	8/5
z_2	-1	(12/5)	4/5	1/5	-1/5	0	0	-1	0	4/5	0	1	-4/5	1/5
y_3	0	13/5	1/5	9/5	1/5	0	0	0	1	1/5	0	0	-1/5	9/5
x_6	0	-3/5	1/5	1/5	-1/5	1	0	0	0	-1/5	0	0	1/5	1/5
$-z$	1	1/5+2/5	8/5	12/5	0	0	0	0	0	2/5	0	0	-2/5	-2/5
$-\varphi$	0	-13/5	1/5	9/5	-1/5	0	1	1	1	-11/5	0	0	11/5	-9/5
z_1	13/12	0	-8/12	19/12	5/12	0	-1	1/12	0	4/12	1-1/12	-4/12	19/12	19/12
x_2	-5/12	1	4/12	1/12	1/12	0	0	5/12	0	14/12	0	1/12	-4/12	1/12
y_3	13/12	0	-8/12	19/12	5/12	0	0	13/12	1	-8/12	0-13/12	8/12	19/12	19/12
x_6	-3/12	0	0	3/12	-3/12	1	0	-3/12	0	0	0	3/12	0	3/12
$-z$	13/12	0	4/12	19/12	29/12	0	0	1/12	0	4/12	0	-1/12	-4/12	-5/12
$-\varphi$	13/12	0	8/12	19/12	-5/12	0	1	-1/12	1	-16/12	0	13/12	16/12	-19/12

So this method will not help us in getting the solution and we have to look for another method. Getting the dual and trying to solve the dual as follows:

$$\begin{aligned}
 (4) \quad & (x_1 - x_2 + 2x_4 + 2x_5 + 2x_6) & (1) \\
 & -x_1 + x_2 + x_3 - 2x_4 - 2x_6 \leq -2 & u_1 \\
 & x_1 - x_4 + x_5 - 4x_6 \leq -1 & u_2 \\
 & 2x_2 + 2x_4 + x_6 \leq 2 & u_3 \\
 & 3x_2 + x_3 - x_4 + x_5 - 5x_6 \leq -1 & u_4 \\
 & x_i \geq 0, \quad i=1, \dots, 6.
 \end{aligned}$$

$$\therefore D \text{ is: Min } (-2u_1 - u_2 + 2u_3 - u_4)$$

$$\begin{aligned}
 \text{s.t.} \quad & -u_1 + u_2 \geq -1 \\
 & u_1 + 2u_3 + 3u_4 \geq -1 \\
 & u_1 + u_4 \geq 0 \\
 & -2u_1 - u_2 + 2u_3 - u_4 \geq 2 \\
 & u_2 + u_4 \geq 2 \\
 & -2u_1 - 4u_2 + u_3 - 5u_4 \geq 2 \\
 & u_i \geq 0, \quad i=1, 2, 3, 4.
 \end{aligned}$$

$$\begin{aligned}
 \therefore \quad & -u_1 + u_2 & -y_1 + z_1 & = 1 \\
 & -u_1 - 2u_3 - 3u_4 + & y_2 & = 1 \\
 & u_1 + u_4 & -y_3 + z_2 & = 0
 \end{aligned}$$

$$-2u_1 - u_2 + 2u_3 - u_4 - y_4 + x_3 = 2$$

$$u_2 + u_4 - y_5 + x_4 = 2$$

$$-2u_1 + u_2 + u_3 - 5u_4 - y_6 + x_5 = 2$$

$$u_i \geq 0, \quad y_j \geq 0, \quad x_s \geq 0, \quad i=1, \dots, 4, \quad j=1, \dots, 6, \quad s=1, \dots, 5.$$

$$\varphi - 7 = 4u_1 + 3u_2 - 3u_3 + 4u_4 + y_1 + y_3 + y_4 + y_5 + y_6.$$

Basis	u_1	u_2	u_3	u_4	y_1	y_2	y_3	y_4	y_5	y_6	z_1	z_2	z_3	z_4	z_5	K
z_1	-1	1	0	0	-1	0	0	0	0	0	1	0	0	0	0	1
y_2	-1	0	-2	-3	0	1	0	0	0	0	0	0	0	0	0	1
z_2	1	0	0	1	0	0	-1	0	0	0	0	1	0	0	0	0
z_3	-2	-1	2	-1	0	0	0	-1	0	0	0	0	1	0	0	2
z_4	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	0	2
z_5	-2	-4	1	-5	0	0	0	0	0	-1	0	0	0	0	1	2
$-z$	-2	-1	2	-1	0	0	0	0	0	0	0	0	0	0	0	0
$-u$	4		-3	4	1	0		1	1	1	0	0	0	0	0	-7
z_1	-1		0	0	-1	0	0	0	0	0	1	0	0	0	0	1
y_2	-3	-1	0	-4	0	1	0	-1	0	0	0	0	1	0	0	3
z_2	1	0	0	1	0	0	-1	0	0	0	0	1	0	0	0	0
u_3	-1	-1/2	1	-1/2	0	0	0	-1/2	0	0	0	0	1/2	0	0	1
z_4	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	0	2
z_5	-1	-1/2	0	-9/2	0	0	0	1/2	0	-1	0	0	-1/2	0	1	1
$-z$	0	0	0	0	0	0	0	1	0	0	0	0	-1	0	0	-2
$-u$	1	3/2	0	5/2	1	0	1	-1/2	1	1	0	0	3/2	0	0	-4
z_1	-1	1	0	0	-1	0	0	0	0	0	1	0	0	0	0	1
y_2	-5	-8	0	-13	0	1	0	0	0	-2	0	0	0	0	2	5
z_2	1	0	0	1	8	0	-1	0	0	0	0	1	0	0	0	0
z_3	-2	-4	1	-5	0	0	0	0	0	-1	0	0	0	0	1	2
z_4	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	0	2
y_4	-2	-7	0	-9	0	0	0	1	0	-2	0	0	-1	0	2	2
$-z$	2	7	0	9	0	0	0	0	0	2	0	0	0	0	-2	-4
$-u$	0	-2	0	-2	1	0	1	0	1	0	0	0	1	0	1	-3
z_2	-1	1	0	0	-1	0	0	0	0	0	1	0	0	0	0	1
y_2	-13	0	0	-13	-8	1	0	0	0	-2	8	0	0	0	2	13
z_2	1	0	0	1	8	0	-1	0	0	0	0	1	0	0	0	0
z_3	-6	0	1	-5	-4	0	0	0	0	-1	4	0	0	0	1	6

Basis	u_1	u_2	u_3	u_4	y_1	y_2	y_3	y_4	y_5	y_6	z_1	z_2	z_3	z_4	z_5	x
z_4	1	0	0	1	1	0	0	0	-1	0	-1	0	0	1	0	1
y_4	-9	0	0	-9	-7	0	0	1	0	-2	7	0	-1	0	2	9
$-z$	9	0	0	9	7	0	0	0	0	2	-7	0	0	0	-2	-11
$-y$	-2	0	0	-2	-1	0	1	0	1	0	2	0	1	0	1	-1
u_2	0	1	0	1	-1	0	-1	0	0	0	1	1	0	0	0	1
y_2	0	0	0	0	-3	1	-13	0	0	-2	8	13	0	0	2	13
u_1	1	0	0	1	0	0	-1	0	0	0	0	1	0	0	0	0
u_3	0	0	1	1	-4	0	-6	0	0	-1	4	5	0	0	1	6
z_4	0	0	0	0	1	0	1	0	-1	0	-1	-1	0	1	0	1
y_4	0	0	0	0	-7	0	-9	1	0	-2	7	9	-1	0	2	9
$-z$	0	0	0	0	7	0	9	0	0	2	-7	-9	0	0	-2	-11
$-y$	0	0	0	0	-1	0	-1	0	1	0	2	2	1	0	1	-1
u_2	0	1	0	1	0	0	0	0	-1	0	0	0	0	1	0	0
y_2	0	0	0	0	0	1	-5	0	-3	-2	0	5	0	8	2	21
u_1	1	0	0	1	0	0	-1	0	0	0	0	1	0	0	0	0
u_3	0	0	1	1	0	0	-2	0	-4	-1	0	2	0	4	1	10
y_1	0	0	0	0	1	0	1	0	-1	0	-1	-1	0	1	0	1
y_4	0	0	0	0	0	0	-2	1	-7	-2	0	2	-1	2	2	16
$-z$	0	0	0	0	0	0	0	0	7	2	0	-2	0	-7	-2	-18
$-y$	0	0	0	0	0	0	0	0	0	0	1	1	1	1	1	0

Since $\varphi = 0$, then there is a permissible solution with $x_s = 0, s = 1, \dots, 5$ and then we have:

B	u_1	u_2	u_3	u_4	y_1	y_2	y_3	y_4	y_5	y_6	K
u_2	0	1	0	1	0	0	-1	0	0	0	2
y_2	0	0	0	0	0	1	-5	0	-8	-2	21
u_1	1	0	0	1	0	0	-1	0	0	0	0
u_3	0	0	1	1	0	0	-2	0	-4	-1	10
y_1	0	0	0	0	1	0	1	0	-1	0	1
y_4	0	0	0	0	0	0	-2	1	-7	-2	16
$-z$	0	0	0	0	0	0	0	0	7	2	-18

But, since we minimizing z and we have non - negative coeffs in the last row so this is the optimum solution and:

$$u_1 = 0$$

$$y_1 = 1$$

$$u_2 = 2$$

$$y_2 = 21$$

$$u_3 = 10$$

$$y_3 = 0$$

$$u_4 = 0$$

$$y_4 = 16$$

$$y_5 = 0$$

$$y_6 = 0$$

$$(5) \text{ Max } (-3x_1 + x_2 - 5x_3)$$

$$\text{s.t.} \quad x_1 + 2x_2 + 5x_3 \geq 2$$

$$x_1 - x_2 \geq -5$$

$$-2x_1 + x_2 - 3x_3 \leq 12$$

$$x_i \geq 0, i=1,2,3.$$

Or

$$\text{Min. } (3x_1 + x_2 + 5x_3)$$

$$\text{s.t.} \quad x_1 + 2x_2 + 5x_3 - y_1 + z_1 = 2$$

$$-x_1 + x_2 + y_2 = 5$$

$$-2x_1 + x_2 - 3x_3 + y_3 = 12; x_i, y_j, z_1 \geq 0.$$

$$\phi - 2 = -x_1 - 2x_2 - 5x_3 + y_1$$

Basis	x_1	x_2	x_3	y_1	y_2	y_3	z_1	K
z_1	1	2	(5)	-1	0	0	1	2
y_2	-1	1	0	0	1	0	0	5
y_3	-2	1	-3	0	0	1	0	12
$-z$	3	1	5	0	0	0	0	0
$-\phi$	-1	-2	-5	1	0	0	0	-2
x_3	2/5	2/5	1	-1/5	0	0	1/5	2/5
y_2	-1	1	0	0	1	0	0	5
y_3	-7/5	11/5	0	-3/5	0	1	3/5	66/5
$-z$	2	-1	0	-1	0	0	-1	-2
$-\phi$	0	0	0	0	0	0	1	0

Basis	x_1	x_2	x_3	y_1	y_2	y_3	z_1	Δ
x_3	1/5	2/5	1	-1/5	0	0		2/5
y_2	-1	1	0	0	1	0		5
y_3	-7/5	11/5	0	-3/5	0	1		66/5
$-z$	2	-1	0	1	0	0		-2

Like problem number 4, we have to look for another method as:

$$(5) \text{ Max } (-3x_1 - x_2 - 5x_3)$$

$$\text{s.t. } x_1 + 2x_2 + 5x_3 \geq 2$$

$$x_1 - x_2 \geq -5$$

$$-2x_1 + x_2 - 3x_3 \leq 12$$

$$x_i \geq 0, i=1,2,3.$$

$$\text{OR } x_1 + 2x_2 + 5x_3 - y_1 + z_1 = 2$$

$$-x_1 + x_2 + y_2 = 5$$

$$-2x_1 + x_2 - 3x_3 + y_3 = 12 \quad ; \quad x_i, y_j, z_1 \geq 0.$$

$$\phi - z = -x_1 - 2x_2 - 5x_3 + y_1$$

Basis	x_1	x_2	x_3	y_1	y_2	y_3	z_1	K
z_1	1	2	5	-1	0	0	1	2
y_2	-1	1	0	0	1	0	0	5
y_3	-2	1	-3	0	0	1	0	12
$-z$	-3	-1	-5	0	0	0	0	0
$-y_1$	-1	-2	-5	1	0	0	0	-2
x_3	1/5	2/5	1	-1/5	0	0	1/5	2/5
y_2	-1	1	0	0	1	0	0	5
y_3	-7/5	11/5	0	-3/5	0	1	3/5	66/5
$-z$	-2	1	0	-1	0	0	1	2
$-y_1$	0	0	0	0	0	0	1	0
x_3	1/5	2/5	1	-1/5	0	0	2/5	1
y_2	-1	1	0	0	1	0	5	0
y_3	-7/5	11/5	0	-3/5	0	1	66/5	6
$-z$	-2	1	0	-1	0	0	2	
x_2	1/2	1	5/2	-1/2	0	0	1	
y_2	-3/2	0	-5/2	1/2	1	0	4	
y_3	-5/2	0	-11/2	1/2	0	1	11	
$-z$	-5/2	0	-5/2	-1/2	0	0	1	

Then

$$x_1 = 0$$

$$y_1 = 0'$$

$$\therefore x_2 = 1$$

$$y_2 = 4$$

$$x_3 = 0$$

$$y_3 = 11$$

$$z = -1$$

$$(6) \quad \text{Max } (x_1 - x_2 + 2x_3) !$$

$$\text{s.t. } 2x_1 - 5x_2 + x_3 = 3$$

$$x_1 - 2x_2 - x_3 = 1$$

$$3x_1 - 5x_2 = 4$$

$$x_i \geq 0, \quad i=1,2,3.$$

OR

$$\text{Max } (x_1 - x_2 + 2x_3) !$$

$$\text{s.t. } 2x_1 - 3x_2 + x_3 + y_1 = 3$$

$$x_1 - 2x_2 - x_3 + y_2 = 1$$

$$3x_1 - 5x_2 + y_3 = 4$$

$$x_i, y_j \geq 0, \quad \varphi = 8 - 6x_1 + 10x_2$$

Basic	x_1	x_2	x_3	y_1	y_2	y_3	K
	2	-3	1	1	0	0	3
y_2	(1)	-2	-1	0	1	0	1
y_3	3	-5	0	0	0	1	4
-3	1	-1	2	0	0	0	0
-4	-6	+10	0	0	-2	0	-8
y_1	0	1	3	1	-2	0	1
x_1	1	-2	-1	0	1	0	1
y_3	0	1	(3)	0	-3	1	1
-2	0	1	3	0	-1	0	-1
-4	0	-2	-6	0	6	0	-2
y_1	0	0	0	1	1	-1	0
x_1	1	-5/3	0	0	0	1/3	4/3
x_3	0	1/3	1	0	-1	1/3	1/3
-3	0	0	0	0	2	-1	-2
-4	0	0	0	0	0	2	0

We get an unfeasibility because of redundancy. So, we have:

$$(6) \quad \text{Max } (x_1 - x_2 + 2x_3)!$$

s.t.

$$2x_1 - 3x_2 + x_3 = 3$$

$$x_1 - 2x_2 - x_3 = 1$$

$$3x_1 - 5x_2 = 4, \quad x_i \geq 0, \quad i=1,2,3.$$

But the third eq. equals the addition of the first two.

∴ The problem becomes:

$$\text{Max } (x_1 - x_2 + 2x_3)!$$

s.t.

$$2x_1 - 3x_2 + x_3 + y_1 = 3$$

$$x_1 - 2x_2 - x_3 + y_2 = 1, \quad x_i \geq 0, \quad y_j \geq 0.$$

$$\varphi = 4 - 3x_1 + 5x_2$$

Basis	x_1	x_2	x_3	y_1	y_2	K
y_1	2	-3	1	1	0	3
y_2	(1)	-2	-1	0	1	1
$-z$	1	-1	2	0	0	0
$-b_1$	-3	5	0	0	0	-4
y_1	0	1	(3)	1	-2	1
x_1	1	-2	-1	0	1	1
$-z$	0	1	3	0	-1	-1
$-b_1$	0	-1	-3	0	3	-1
x_3	0	1/3	1	1/3	-2/3	1/3
x_1	1	-5/3	0	1/3	1/3	4/3
$-z$	0	0	0	-1	1	-2
$-b_1$	0	0	0	1	1	0
x_3	0	1/3	1			1/3
x_1	1	-5/3	0			4/3
$-z$	0	0	0			-2

so we have a
permissible
solution

But the last row is zero, so the solution is:

$$x_1 = \frac{4}{3}$$

$$x_2 = 0$$

$$x_3 = \frac{1}{3}$$

$$(7) \text{ Max } (3x_1 - 6x_2 + x_4 + 2x_6)$$

s.t.

$$x_1 + x_2 + x_3 + 2x_4 - 6x_6 \leq 2$$

$$x_1 - x_4 + x_5 - 4x_6 \leq 1$$

$$2x_2 + 2x_4 + x_6 \leq 2$$

$$2x_1 + 3x_2 + x_3 + 3x_4 + x_5 - 15x_6 \leq 5$$

$$x_i \geq 0, \quad i=1, \dots, 6$$

OR

$$x_1 + x_2 + x_3 + 2x_4 - 6x_6 + y_1 = 2$$

$$x_1 - x_4 + x_5 - 4x_6 + y_2 = 1$$

$$2x_2 + 2x_4 + x_6 + y_3 = 2$$

$$2x_1 + 3x_2 + x_3 + 3x_4 + x_5 - 15x_6 + y_4 = 5$$

$$x_i, y_j \geq 0, \quad i=1, \dots, 6, \quad j=1, \dots, 4.$$

	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	K
Basis	5	-6	0	1	0	2	0	0	0	0	
y_1	1	1	1	2	0	-6	1	0	0	0	2
y_2	(1)	0	0	-1	1	-4	0	1	0	0	1
y_3	0	2	0	2	0	1	0	0	1	0	2
y_4	2	3	1	3	1	-15	0	0	0	1	5
f	0	0	0	0	0	0	0	0	0	0	0
$z-f$	[3]	-6	0	1	0	2	0	0	0	0	
y_1	0	1	1	3	-1	-2	1	-1	0	0	1
x_1	1	0	0	-1	1	-4	0	1	0	0	1
y_3	0	2	0	2	0	(1)	0	0	1	0	2
y_4	0	3	1	5	-1	-7	0	-2	0	1	3
f	3	0	0	-3	3	-12	0	3	0	0	3
$z-f$	0	-6	0	4	-3	[14]	0	-3	0	0	
y_1	0	5	1	7	-1	0	1	-1	2	0	5
x_1	1	8	0	7	1	0	0	1	4	0	9
x_6	0	2	0	2	0	1	0	0	1	0	2
y_4	0	17	1	19	-1	0	0	-2	7	1	17
f	3	28	0	25	3	2	0	3	14	0	31
$z-f$	0	-34	0	-24	-3	0	0	-3	-14	0	

Since the last row is non-positive $\Rightarrow$ the optimum.

$$\begin{aligned}
 \therefore x_1 &= 9 & y_1 &= 5 & \text{and } z &= 31 \\
 x_2 &= 0 & y_2 &= 0 \\
 x_3 &= 0 & y_3 &= 0 \\
 x_4 &= 0 & y_4 &= 17 \\
 x_5 &= 0 \\
 x_6 &= 2
 \end{aligned}$$

Where as :

$$(8) \quad \text{Max } (3x_1 - 2x_2 + 2x_3 + 2x_4) = P$$

$$\text{s.t. } 2x_1 - x_2 + x_3 + x_4 \leq 2$$

$$-3x_1 + 2x_2 - 2x_3 - x_4 \leq 1$$

$$-2x_1 + 2x_2 - x_3 - x_4 \leq 2, \quad x_i \geq 0, i=1, \dots, 4.$$

(R

OR min

$$(2u_1 + u_2 + 2u_3)$$

= D

s.t.

$$-2u_1 - 3u_2 - 2u_3 \geq 3$$

$$-u_1 + 2u_2 + 2u_3 \geq -2$$

$$u_1 - 2u_2 - u_3 \geq 2$$

$$u_1 - u_2 - u_3 \geq 2, \quad u_j \geq 0, j=1, 2, 3.$$

i.e.

$$2u_1 - 3u_2 - 2u_3 - \zeta_1 + \eta_1 = 3$$

$$u_1 - 2u_2 - 2u_3 + \zeta_2 = 2$$

$$\text{OP } u_1 - 2u_2 - u_3 - \zeta_3 + \eta_2 = 2$$

$$u_1 - u_2 - u_3 - \zeta_4 + \eta_3 = 2, \quad u_i, \zeta_j, \eta_s \geq 0.$$

$$\varphi = 7 - 4u_1 + 6u_2 + 4u_3 + \zeta_1 + \zeta_3 + \zeta_4.$$

Basis	u_1	u_2	u_3	ξ_1	ξ_2	ξ_3	ξ_4	η_1	η_2	η_3	K
η_1	(2)	-3	-2	-1	0	0	0	1	0	0	3
ξ_2	1	-2	-2	0	1	0	0	0	0	0	2
η_2	1	-2	-1	0	0	-1	0	0	1	0	2
η_3	1	-1	-1	0	0	0	-1	0	0	1	2
-2	2	1	2	0	0	0	0	0	0	0	0
-4	-4	6	4	1	0	1	1	0	0	0	-7
u_1	1	-3/2	-1	-1/2	0	0	0	1/3	0	0	3/2
ξ_2	0	-1/2	-1	1/2	1	0	0	-1/2	0	0	1/2
η_2	0	-1/2	0	1/2	0	-1	0	-1/2	1	0	1/2
η_3	0	1/2	0	1/2	0	0	-1	-1/2	0	1	1/2
-2	0	4	4	1	0	0	0	-1	0	0	-3
-4	0	0	0	-1	0	0	1	2	0	0	-1
u_1	1	-2	-1	0	0	-1	0	0	1	0	2
ξ_2	0	0	-1	0	1	1	0	0	-1	0	0
ξ_1	0	-1	0	1	0	-2	0	-1	2	0	1
η_3	0	1	0	0	0	(1)	-1	0	-1	1	0
-2	0	5	4	0	0	2	0	0	-2	0	-4
-4	0	-1	0	0	0	-2	1	1	2	0	0
u_1	1	-1	-1	0	0	0	-1	0	0	1	2
ξ_2	0	-1	-1	0	1	0	(1)	0	0	-1	0
ξ_1	0	1	0	1	0	0	-2	-1	0	2	1
ξ_3	0	1	0	0	0	1	-1	0	-1	1	0
-2	0	3	4	0	0	0	2	0	0	-2	-4
-4	0	1	0	0	0	0	-1	1	0	2	0

Basis	u_1	u_2	u_3	τ_1	τ_2	τ_3	τ_4	η_1	η_2	η_3	K
	1	0	-2	0		0	0	0	0		2
	0	-1	-1	0	1	0	1	0	0	-1	0
ξ_1	0	-1	-2	1		0	0	-1	0		1
ξ_3	0	0	-1	0		1	0	0	-1		0
$-z$	0	5	6	0		0	0	0	0		-4
$-q$	0	0	-1	0		0	0	1	0		0

We did not get a feasible point. Thus, we move to.

$$(8) \quad \text{Max } (3x_1 - 2x_2 + 2x_3 + 2x_4)$$

$$\text{s.t.}, \quad 2x_1 - x_2 + x_3 + x_4 \leq 2$$

$$-3x_1 + 2x_2 - 2x_3 - x_4 \leq 1$$

$$-2x_1 + 2x_2 - x_3 - x_4 \leq 2, \quad x_i \geq 0, \quad i=1, \dots, 4.$$

OR:

$$\text{Max } (3x_1 - 2x_2 + 2x_3 + 2x_4)$$

$$\text{s.t.}, \quad 2x_1 - x_2 + x_3 + x_4 + y_1 = 2$$

$$-3x_1 + 2x_2 - 2x_3 - x_4 + y_2 = 1$$

$$-2x_1 + 2x_2 - x_3 - x_4 + y_3 = 2, \quad x_i \geq 0, \quad y_j \geq 0$$

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	K
	3	-2	2	2	0	0	0	
y_1	(2)	-1	1	1	1	0	0	2
y_2	-3	2	-2	-1	0	1	0	1
y_3	-2	+2	-1	-1	0	0	1	2
z	0	0	0	0	0	0	0	0
$z-f$	[3]	-2	2	2	0	0	0	
x_1	1	$-\frac{1}{2}$	($\frac{1}{2}$)	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
y_2	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{3}{2}$	1	0	4
y_3	0	1	0	0	1	0	1	4
z	3	$-\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}$	0	0	3
$z-f$	0	$-\frac{1}{2}$	[$\frac{1}{2}$]	$\frac{1}{2}$	$-\frac{3}{2}$	0	0	
x_3	2	-1	1	1	1	0	0	2
y_2	1	0	0	1	2	1	0	5
y_3	0	1	0	0	1	0	1	4
z	4	-2	2	2	2	0	0	4
$z-f$	-1	0	0	0	-2	0	0	

I.e. we reached the maximum with:

$$\begin{array}{lll}
 x_1 = 0 & y_1 = 0 & z = 4 \\
 x_2 = 0 & y_2 = 5 & \\
 x_3 = 2 & y_3 = 4 & \\
 x_4 = 0 & &
 \end{array}$$

OR

x_4	2	-1	1	1	1	0	0	2
y_2	-1	1	-1	0	1	1	0	3
y_3	0	1	0	0	1	0	1	4
z	4	-2	2	2	2	0	0	4
$z-f$	-1	0	0	0	-2	0	0	

is another maximal point with:

$$x_1 = 0$$

$$y_1 = 0$$

$$z = 4$$

$$x_2 = 0$$

$$y_2 = 3$$

$$x_3 = 0$$

$$y_3 = 4$$

$$x_4 = 2$$

9. Find the value of λ which minimize $u \geq 0$.

$$\begin{aligned} f(x) &= x + y + z - u \\ 2x - 3y + 4z - u &= 3 \\ 3x + y - 3z + 2u &\leq 0 \\ -x + y - z - \lambda u &\geq 1 \end{aligned} \quad ; x, y, z, u, \lambda \geq 0.$$

$$\begin{aligned} \text{OR} \quad 2x - 3y + 4z - u + \eta_1 &= 3 \\ 3x + y - 3z + 2u + \xi_1 &= 0 \end{aligned}$$

$$-x + y - z - \lambda u - \xi_2 + \eta_2 = 1$$

$$\begin{aligned} \text{Let } \varphi = \eta_1 + \eta_2 &= 3 - 2x + 3y - 4z + u - \eta_1 + 1 + x - y + z + \lambda u + \xi_2 \\ &= 4 - x + 2y - 3z + (1 + \lambda)u + \xi_2 \end{aligned}$$

Basis	x	y	z	u	ξ_1	ξ_2	η_1	η_2	Const.
η_1	2	-3	(4)	-1	0	0	1	0	3
ξ_1	3	1	-3	2	1	0	0	0	0
η_2	-1	1	-1	$-\lambda$	0	-1	0	1	1
-f	1	1	1	-1	0	0	0	0	0
$-\varphi$	-1	2	(-3)	$1 + \lambda$	1	1	0	0	Min.
z	2/4	-3/4	1	-1/4	0	0	1/4	0	3/4
ξ_1	18/4	-5/4	0	5/4	1	0	3/4	0	9/4
η_2	-2/4	(1/4)	0	$-\lambda - \frac{1}{4}$	0	-1	1/4	1	7/4
-f	2/4	7/4	0	-3/4	0	0	-1/4	0	-3/4
$-\varphi$	2/4	(-1/4)	0	$+\frac{1}{4} + \lambda$	0	1	+3/4	0	-7/4

Basis	x	y	z	w	ξ_1	ξ_2	η_1	η_2	Const.
z	-1	0	1	$-1-3\lambda$	0	-3	1	3	6
ξ_1		0	0	-5λ	1	-5	2	5	11
y		1	0	$-4\lambda-1$	0	-4	1	4	7
$-f$	4	0	0	$1+7\lambda$	0	7	-2	-7	-13
$-q$	0	0	0	0	0	0	1	1	0

Basis	x	y	z	w	ξ_1	ξ_2	Const.
z	-1	0	1	$-1-3\lambda$	0	-3	6
ξ_1	(2)	0	0	-5λ	1	-5	11
y	-2	1	0	$-4\lambda-1$	0	-4	7
$-f$	4	0	0	$1+7\lambda$	0	7	-13
							Max
z	0	0	1	$-1-\frac{11}{2}\lambda$	1/2	-11/2	23/2
x	1	0	0	$-5\lambda/2$	1/2	-5/2	11/2
y	0	1	0	$-1-9\lambda$	1	-9	18
$-f$	0	0		$1+17\lambda$	-2	17	-35

Which means that:

$$x = \frac{-11}{2}, y = 18, z = \frac{23}{2}, w = 0$$

and $f = 35$

Now by getting the dual we have:

$$P: f = x + y + z - u$$

$$s.t. \quad 2x - 3y + 4z - u = 3$$

$$3x + y - 3z + 2u \leq 0$$

$$+ x - y + z + \lambda u \leq 1$$

$$y, z, u, \lambda \geq 0$$

$$D: \min g = 3\xi + \zeta$$

$$s.t. \quad 2\xi + 3\eta + \zeta \geq 1$$

$$-3\xi + \eta - \zeta \geq 1$$

$$4\xi - 3\eta + \zeta \geq 1$$

$$-\xi + 2\eta + \lambda\zeta \geq -1$$

$$\xi, \eta, \zeta \geq 0.$$

Or the canonical form is:

$$\min g = 3\xi + \zeta$$

s.t.

$$2\xi + 3\eta + \zeta - x_1 + y_1 = 1$$

$$-3\xi + \eta - \zeta - x_2 + y_2 = 1$$

$$4\xi - 3\eta + \zeta - x_3 + y_3 = 1$$

$$\xi - 2\eta - \lambda\zeta + x_4 = 1$$

$$\xi, \eta, \zeta, x_1, y_1 \geq 0.$$

$$\begin{aligned} \psi &= \sum_{j=1}^3 y_j = 1 - 2\xi - 3\eta - \zeta + x_1 + 1 + 3\xi - \eta + x_2 + 1 - 4\xi + 3\eta - \zeta + x_3 \\ &= 3 - 3\xi - \eta - \zeta + x_1 + x_2 + x_3 \end{aligned}$$

and we have

Basis	ξ	η	ξ	x_1	x_2	x_3	x_4	y_1	y_2	y_3	Const.
y_1	2	3	1	-1	0	0	0	1	0	0	1
y_2	-3	1	-1	0	-1	0	0	0	1	0	1
y_3	(4)	-3	1	0	0	-1	0	0	0	1	1
x_4	1	-2	$-\lambda$	0	0	0	1	0	0	0	1
-g	3	0	1	0	0	0	0	0	0	0	0
- ψ	-3	-1	-1	1	1	1	0	0	0	0	-3 min
y_1	0	(18/4)	2/4	-1	0	+2/4	0	1	0	-2/4	2/4
y_2	0	-5/4	-1/4	0	-1	-3/4	0	0	1	3/4	7/4
ξ	1	-3/4	1/4	0	0	-1/4	0	0	0	1/4	1/4
x_1	0	-5/4	$\lambda = \frac{1}{4}$	0	0	1/4	1	0	0	-1/4	3/4
-g	0	9/4	1/4	0	0	3/4	0	0	0	-3/4	-3/4
- ψ	0	-13/4	-1/4	1	1	1/4	0	0	0	3/4	-9/4
η	0	1	2/18	-4/18	0	2/18	0	4/18	0	-2/18	2/18
y_2	0	0	-2/18	-5/18	-1	-11/18	0	5/18	1	11/18	34/18
ξ	1	0	6/18	-3/18	0	-3/18	0	3/18	0	3/18	6/18
x_1	0	0	$\lambda = \frac{2}{18}$	-5/18	0	7/18	1	5/18	0	-7/18	16/18
-g	0	0	0	9/18	0	9/18	0	-9/18	0	-9/18	-18/18
- ψ	0	0	2/18	5/18	1	11/18	0	13/18	0	7/18	-34/18

another trial, we have

$$f = x + y + z - u$$

$$x_1 \quad 3x + y - 3z + 2u \leq 0$$

$$x_2 \quad x - y + z + \lambda u \leq 1$$

$$x_3 \quad 2x - 3y + 4z - u \leq 3$$

$$x_4 \quad -2x + 3y - 4z + u \leq -3$$

The dual is :

$$\min g = x_2 + 3x_3 - 3x_4$$

s.t.

$$3x_1 + x_2 + 2x_3 - 2x_4 \geq 1$$

$$x_1 - x_2 - 3x_3 + 3x_4 \geq 1$$

$$-3x_1 + x_2 + 4x_3 - 4x_4 \geq 1$$

$$2x_1 + \lambda x_2 - x_3 + x_4 \geq -1$$

$$-2x_1 - \lambda x_2 + x_3 - x_4 \leq 1$$

By eliminating the fourth constraint (because of redundancy), we get :

$$3x_1 + x_2 + 2x_3 - 2x_4 - y_1 + z_1 = 1$$

$$x_1 - x_2 - 3x_3 + 3x_4 - y_2 + z_2 = 1$$

$$-3x_1 + x_2 + 4x_3 - 4x_4 - y_3 + z_3 = 1$$

$$-2x_1 - \lambda x_2 + x_3 - x_4 + y_4 = 1$$

$$\psi = \sum_{j=1}^3 z_j$$

$$= x_1 + x_2 + 3x_3 - 3x_4 - y_1 - y_2 + y_3 + \psi = 3$$

$$\psi - 3 = -x_1 - x_2 - 3x_3 + 3x_4 + y_1 + y_2 + y_3$$

and it appears as :

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	z_1	z_2	z_3	Const.
z_1	3	1	2	-2	-1	0	0	0	1	0	0	1
z_2	1	-1	-3	3	0	-1	0	0	0	1	0	1
z_3	-3	1	(4)	-4	0	0	-1	0	0	0	1	1
y_4	-2	$-\lambda$	1	-1	0	0	0	1	0	0	0	1
-g	0	1	3	-3	0	0	0	0	0	0	0	0
$-\psi$	-1	-1	(-3)	+3	1	1	1	0	0	0	0	-3min
z_1	(18/4)	2/4	0	0	-1	0	2/4	0	1	0	-2/4	2/4
z_2	-5/4	-1/4	0	0	0	-1	-3/4	0	0	1	3/4	7/4
x_3	-3/4	1/4	1	-1	0	0	-1/4	0	0	0	1/4	1/4
y_4	-5/4	$-\lambda - \frac{1}{4}$	0	0	0	0	1/4	1	0	0	-1/4	3/4
-g	9/4	1/4	0	0	0	0	3/4	0	0	0	-3/4	-3/4
$-\psi$	(-13/4)	-1/4	0	0	1	1	1/4	0	0	0	3/4	-9/4
x_1	1	2/18	0	0	-4/18	0	2/18	0	4/18	0	-2/18	2/18
z_2	0	-2/18	0	0	-5/18	-1	-11/18	0	-5/18	1	11/18	34/18
x_3	0	6/18	1	-1	-3/18	0	-3/18	0	3/18	0	3/18	6/18

Basis	x_1	x_2	x_3	x_4	y_1	y_2	y_3	y_4	z_1	z_2	z_3	Const.
4	0	3	0	0	-5/18	0	7/18	1	5/18	0	-7/18	16/18
g	0	0	0	0	9/18	0	9/18	0	-9/18	0	-9/18	-18/18
y	0	2/18	0	0	5/18	1	11/18	0	13/18	0	7/18	-34/18

Which gives the same result as before.

10. Get the dual of the following problems :

$$a^{\circ} P: \min (2x_1 - 3x_2 + x_3 - 4x_4)!$$

$$s.t. \quad x_1 - x_2 + 3x_3 - x_4 = 2$$

$$2x_1 + x_2 - 5x_3 = -3, \quad x_i \geq 0, \quad i = 1, \dots, 4$$

$$So, \text{ we have } \max (-2x_1 + 3x_2 - x_3 + 4x_4)!$$

$$D: \min (2u_1 - 3u_2) ! \Rightarrow \max (-2u_1 + 3u_2)$$

$$s.t. \quad u_1 + 2u_2 \geq -2$$

$$-u_1 + u_2 \geq 3$$

$$3u_1 - 5u_2 \geq -1$$

$$-u_1 \geq 4$$

$$b^{\circ} P: \max (-x_1 + 3x_2 + 4x_3 - 5x_4 + 2x_5)!$$

$$s.t. \quad 7x_1 - 3x_2 + 5x_3 - x_4 + 2x_5 = -1$$

$$8x_1 + x_2 - x_3 + 7x_4 + 3x_5 = 3$$

$$2x_1 + 6x_2 + 6x_3 - 3x_4 - 4x_5 = 9, x_i \geq 0, i=1,2,3,4.$$

$$D: \min (-u_1 + 3u_2 + 9u_3)$$

$$s.t. \quad 7u_1 + 8u_2 + 2u_3 \geq -1$$

$$-3u_1 + u_2 + 6u_3 \geq 3$$

$$5u_1 - u_2 + 6u_3 \geq 4$$

$$-u_1 + 7u_2 - 3u_3 \geq -5$$

$$2u_1 + 3u_2 - 4u_3 \geq 2$$

$$c^0 P: \max (x_1 + 3x_2 + 2x_3 - 3x_5 - x_6)$$

$$s.t. \quad 2x_1 + 2x_2 + x_3 + x_4 + 2x_5 + x_6 = 1$$

$$4x_1 + 3x_2 - x_3 - 2x_4 - x_5 + 2x_6 = 1, x_i \geq 0, i=1, \dots, 6.$$

$$D: \min (u_1 + u_2)$$

$$s.t. \quad 2u_1 + 4u_2 \geq 1$$

$$2u_1 + 3u_2 \geq 3$$

$$u_1 - 1u_2 \geq 2$$

$$u_1 - 2u_2 \geq 0$$

$$2u_1 - u_2 \geq -3$$

$$u_1 + 2u_2 \geq -1$$

$$d^0 \min (x_1 + 3x_2 - x_3 + 2x_4 - x_5)$$

$$\text{or } P: \max (-x_1 - 3x_2 + x_3 - 2x_4 + x_5)$$

$$s.t. \quad 2x_1 - 2x_2 + x_3 - 4x_4 + x_5 - 4x_6 \leq 3$$

$$5x_1 - 4x_2 - 6x_3 + 3x_4 + 2x_5 + x_6 \leq 5$$

$$x_1 + x_2 - 8x_3 + 5x_4 - 2x_5 + 7x_6 = 2, x_i \geq 0, i=1, \dots, 6$$

$$D: \max (3u_1 + 5u_2 + 2u_3)$$

$$\text{s.t. } 2u_1 + 5u_2 + 1u_3 \geq -1$$

$$-2u_1 - 4u_2 + u_3 \geq -3$$

$$u_1 - 6u_2 - 8u_3 \geq +1$$

$$-4u_1 + 3u_2 + 5u_3 \geq -2$$

$$u_1 + 2u_2 - 2u_3 \geq +1$$

$$-4u_1 + u_2 + 7u_3 \geq 0, u_1 \geq 0, u_2 \geq 0.$$

$$e^0 P: \max (2x_1 + x_2 - 7x_3 + 4x_4 - 5x_5 - x_6)$$

$$\text{s.t. } 7x_1 - 6x_2 - 2x_3 + 8x_4 + 5x_5 + 3x_6 \leq 5$$

$$3x_1 - 3x_2 + x_3 - 4x_4 + 2x_5 - 2x_6 = 10$$

$$2x_1 + 8x_2 - 6x_3 - 5x_4 - 7x_5 + 2x_6 = 2, x_i \geq 0, i=1, \dots, 6.$$

$$D: \min (5u_1 + 10u_2 + 2u_3)$$

$$\text{s.t. } 7u_1 + 3u_2 + 2u_3 \geq 2$$

$$-6u_1 - 3u_2 + 8u_3 \geq 1$$

$$-2u_1 + u_2 - 6u_3 \geq -7$$

$$8u_1 - 4u_2 - 5u_3 \geq 4$$

$$5u_1 + 2u_2 - 7u_3 \geq -5$$

$$3u_1 - 2u_2 + 2u_3 \geq -1, \quad u_1 \geq 0.$$

$$f^0 \quad \max (3x_1 - 2x_2 + 4x_3)$$

$$\text{s.t.} \quad 2x_1 + x_2 - x_4 \leq 4$$

$$3x_1 - 2x_2 + x_3 \leq 1$$

$$5x_1 + 4x_2 - 6x_4 \leq 2, \quad x_i \geq 0, i=1, \dots, 4.$$

$$D: \min (4u_1 + u_2 + 2u_3)$$

$$\text{s.t.} \quad 2u_1 + 3u_2 + 5u_3 \geq 3$$

$$u_1 - 2u_2 + 4u_3 \geq -2$$

$$u_2 \geq 4$$

$$-u_1 - 6u_3 \geq 0, \quad u_j \geq 0, j=1, \dots, 3.$$

$$g^0 \quad P: \max (-2x_1 + 5x_2 - 6x_3 - 4x_4)$$

$$\text{s.t.} \quad x_1 - x_2 - 7x_3 + 5x_4 \leq 9$$

$$2x_1 + 3x_2 + x_3 - 6x_4 \leq 7$$

$$x_1 + x_2 - 4x_3 - 3x_4 \leq 4$$

$$5x_1 - 6x_2 + 3x_3 + 6x_4 \leq 5, \quad x_i \geq 0, i=1, \dots, 4$$

$$D: \min (9u_1 + 7u_2 + 4u_3 + 5u_4)$$

$$\begin{aligned} \text{s.t.} \quad & u_1 + 2u_2 + u_3 + 5u_4 \geq -2 \\ & -u_1 + 3u_2 + u_3 - 6u_4 \geq 5 \\ & -7u_1 + u_2 - 4u_3 + 3u_4 \geq -6 \\ & 5u_1 - 6u_2 - 3u_3 + 6u_4 \geq -4, \quad u_j \geq 0, \quad j=1, \dots, 4. \end{aligned}$$

$$h^{\circ} \text{ P: } \min (x_1 - x_2 + 3x_3)$$

$$\text{s.t. } 2x_1 - 5x_2 + x_3 \leq 4$$

$$-x_1 + 3x_2 - 2x_3 \leq 9, \quad x_i \geq 0, \quad i = 1, 2, 3.$$

$$\text{or } \max (-x_1 + x_2 - 3x_3)$$

$$D: \min (4u_1 + 9u_2) \Rightarrow \max (-4u_1 - 9u_2)$$

$$\text{s.t. } 2u_1 - u_2 \geq -1$$

$$-5u_1 + 3u_2 \geq 1$$

$$u_1 - 2u_2 \geq -3, \quad u_j \geq 0, \quad j=1, 2.$$

$$i^{\circ} \text{ P: } \min (2x_1 - 3x_2 - 4x_3)$$

$$\text{s.t. } -3x_1 + 5x_2 - x_3 \leq 6$$

$$2x_1 - x_2 + 6x_3 \leq 5$$

$$x_1 + x_2 + 4x_3 \leq 3$$

$$-4x_1 - 2x_2 - 7x_3 \leq 1, \quad x_i \geq 0, \quad i=1, 2, 3.$$

$$\text{or max } (-2x_1 + 3x_2 - 4x_3) ?$$

$$D : \min (6u_1 + 5u_2 + 3u_3 + u_4)$$

$$\text{s.t } -3u_1 + 2u_2 + u_3 - 4u_4 \geq -2$$

$$5u_1 - u_2 + u_3 - 2u_4 \geq 3$$

$$-u_1 + 6u_2 + 4u_3 - 7u_4 \geq -4, \quad u_j \geq 0, \quad j=1, \dots, 4.$$

