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BOUNDNESS OF PER CAPITA CONSUMPTION STREAM THROUGH TIME

by

Dr. A. Qayum

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CAIRO
3 MOHAMED MAZHAR - ZAMALEK

Boundedness of Per Capita Consumption Stream Through Time

Some recent analyses of the nature of economic stagnation have resulted into findings that suggest that in the absence of technical progress and population control an economy is characterised by the 'impossibility of achieving sustained and unbounded increases in its per capita consumption stream through time'.[1] An important implication of these findings is very pertinent to the capital-short underdeveloped economies, viz., if the above holds true then capital accumulation by itself cannot release these economies from low per capita consumption trap. The purpose of this article is first to present a simple exposition of this so called 'impossibility theorem'.[2] Secondly, an attempt will be made to show how far this theorem really holds. Thirdly some comments will be given as to how the whole discussion is of academic nature only as far as the present-day underdeveloped countries are concerned. Finally the roles of technical progress and population control will be discussed.

I

The 'impossibility theorem' hinges on two crucial assumptions. First, the production function must be subject to diminishing returns to factors, particularly to capital per unit of labour. That is, total product must increase at a diminishing rate as capital per unit of labour increases. This condition is satisfied if the production function is linear and homogeneous, and we shall restrict ourself to this function only in this note. Hence let $Q = f(K,L)$ be the linear homogeneous production function, so that $Q^{*} = Q/_{L} = f(\frac{K}{L}, 1) = f(K^{*}, 1)$ is the output per worker. Then

$$(1) \qquad \frac{dQ^{*}}{dK^{*}} > 0 \;;\; \frac{d^{2}Q^{*}}{dK^{*2}} < 0 \qquad \text{where} \quad K^{*} = \frac{K}{L}$$

The second inequality in (1) implies that the production function is strictly

1)John C.H.Fei: "Towards a Theory of Stagnation" presented in the First World Econometric Conference held at Rome in September 1965.

2)For the sake of brevity, we shall call the above mentioned finding "the impossibility theorem" hereinafter.

convex. When the production function is convex but not strictly, the impossibility theorem may not hold as shown below.

The other assumption is that the rate of growth of capital stock is constant at a <u>positive</u> level. If the rate of growth of capital is zero, the theorem will not hold true and if it is variable, there will not be a unique upper bound on the level of per capita consumption. From a given point of time, the lower the constant rate of growth of capital, the higher the upper bound on the rate of per capita consumption. These aspects will be briefly described below.

Once, these restrictive assumptions are made, the derivation of the impossibility theorem is rather straight-forward. Let total consumption

$$C = f\ (K,L) - \eta K \tag{2}$$

where η is the rate of growth of capital which is assumed to be constant. Then,

$$C^{*} = f\ (K^{*},1) - \eta K^{*} \quad , \quad C^{*} = C/_{L} \tag{3}$$

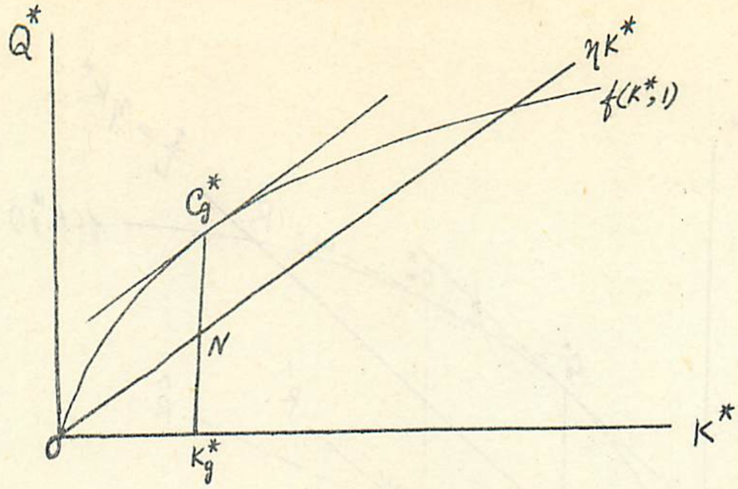
Therefore, the necessary condition for the maximisation of the per capita consumption is

$$\frac{dC^{*}}{dK^{*}} = 0$$

or

$$\frac{df(K^{*},1)}{dK^{*}} = \eta \tag{4}$$

The sufficient condition is satisfied by dint of assumptions (1). And as f is linear homogeneous by assumption, there is only one value of K^{*} which satisfies (4). Hence, the maximum value of C^{*} can be obtained by substituting (4) in (3). The situation is well-depicted by the following diagram.

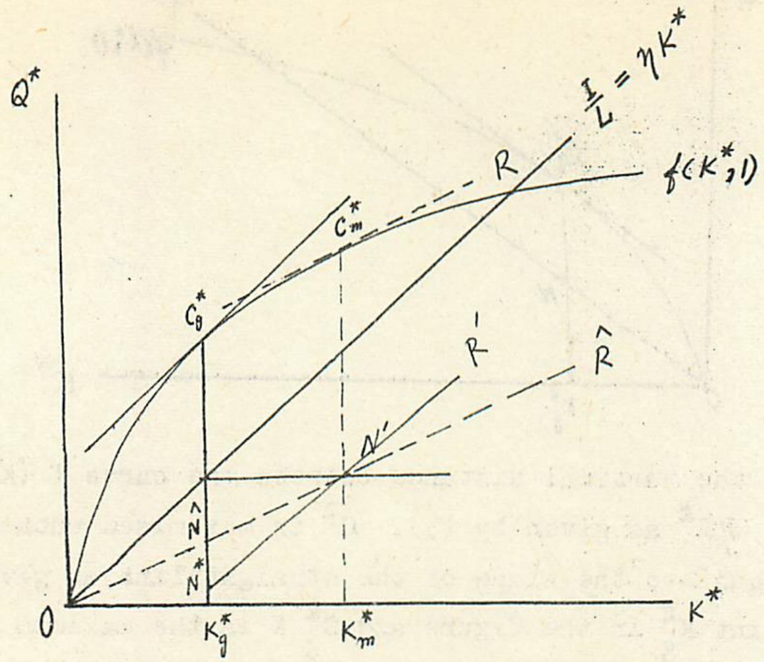


In fig. 1, C^* is the vertical distance between the curve f $(K^*,1)$ and the straight line ηK^* as given by (3). C^* is maximised where the slope of the curve is equal to the slope of the straight line as given by (4). This happen at point K_g^* in the figure and C_g^*N is the maximum value of consumption that can be obtained given the value of η and the production function f. So long as η is positive, and f $(K^*,1)$ strictly convex, the straight line ηK^* must intersect the curve f $(K^*,1)$, so that the area between the two must be bounded and hence there must be an upper limit of the distance between the two resulting into the boundness of the per capita consumption. If $f(K^*,1)$ is not strictly connex, then it can be a straight line, and in that case, it can be easily seen that per capita consumption will be unbounded, when its slope is larger than that of ηK^*. When the slope of ηK^* is larger than that of $f(K^*,1)$ (in case it is a straight line), the per capita consumption will be maximum at zero level. This then is 'the impossibility theorem'.[1)]

II

Under the strict assumptions of the theorem, it is unassailable. But the theorem may be suspected to imply that flow of foreign capital, if unaccompanied by technical progress of any sort, is not going to do any permanent good to an economy if it is already operating at K_g^*.

1) The Theorem can be regarded as a corollary of the celebrated article of R.M. Solow, "A Contribution to the Theory of Economic Growth", Quarterly Journal of Economics 1956.



Let us see what happens if an underdeveloped country receives some capital from outside (in form of grant or aid), so that its K^* is moved from OK^*_g to OK^*_m. As $f(K^*,1)$ is an increasing function of K^*, $C^*_m K^*_m$ must be greater than $C^*_g K^*_g$. $NK^*_g = N'K^*_m$ for $\frac{I}{L} = \frac{I}{K} \cdot \frac{K}{L}$ before the receipt of foreign capital $= \frac{I}{K+M} \times \frac{K+M}{L}$ after the receipt of foreign capital. Hence $C^*_m N' > C^*_g N$ in all cases. Hence an increase in K^* brought about by a flow of foreign capital can raise the upper bound on C^*. And as $f(X^*,1)$ is an increasing function of K^* and infinitum by assumption, C^* can be raised to any level (even in the absence of technical progress) by acquisition of capital, and that it has no upper bounds if this is possible.

But one can object that this will happen in the very short run. In the long run, capital will be routinised, and the economy may again revert back to point C^*_g. I think it may happen in the long run, especially when the population growth λ , remains the same, and $\lambda > \eta$, so that K^* will move to the left to K^*_g. Now the crucial element to consider is how N' moves to the left; along N'N or along $N'N^*$? One would probably say along N'N and he would be right if $\frac{I}{L}$ is maintained as at $K^*_{m'}$ during this transition from K^*_m to K^*_g, where I stands for investment. An alternative line may be that, though at N', η has been reduced from η to η' ($\eta' < \eta$)

when K_m^* moves towards the left N should move along $N'N^*$ so that $\frac{I}{L}$ declines in the same proportion as if N moved towards O. For at K_m^* and K_g^*, the amount of investment is the same, and if growth of labour outstrips growth of capital, so that $\lambda > \eta$; $\frac{I}{L}$ should decline in the same ratio as it would have done in the absence of capital from abroad. In the case, of course, the assertion of the theoren will be proved. In the second case it can be seen that $C_g^* N^* > C_g^* N$ and the assertion will not hold true. The economy will be settled at a point where K^* reaches K_g^*, and it can be seen now that the upper bound has been changed.

Leaving apart these two extremes, we can now envisage a thrid course. This course is that which leaves in successive periods the amount of total investment with and without the use of foreign capital intact and unchanged. This is obtained if we move along N'O. It can be easily demonstrated. At points K_m^* and K_g^*, total saving or total investment is the same, for the heights NK_g^* and $N'K_m^*$ measure $\frac{I}{L}$, but L being the same in both cases, they represent equal amount of investment, so that

$$NK_g^* = \eta . K = \eta' (K+M) = N'K_m^*$$

$$\therefore \quad \eta = \eta' (1 + \frac{M}{K})$$

In a period, t periods hence, the total capital will be in the absence of injection of foreign capital

$$K (1 + \eta)^t = K\left\{ 1 + \eta' (1 + \frac{M}{K})\right\}^t$$

$$= K\left\{ 1 + t\eta' (1 + \frac{M}{K})\right\} \quad \text{approx.}$$

$$= K + tK\eta' + t\eta' M$$

In the same period total capital will be after the introduction of the foreign capital in the initial period

$$(K + M) (1 + \eta')^t = (K + M) (1 + t\eta') \quad \text{approx.}$$

$$= K + t\eta' K + \eta' tM + M$$

In both cases the two quantities are the same except that in the latter case there is an additional M, but this is what was introduced in the initial period from abraod. Now if N' moves along N'O, the consumption per head at K_g^* will be $C_g^* \hat{N} > C_g^* N$. $C_g^* \hat{N}$, of course, is not the maximum consumption along ON'. Again the upper boundary has been outstripped.

However in most of the so-called traditional economies, rate of growth of capital is slightly higher than the rate of growth of population $\eta > \lambda$ except perhaps in the case of Indonesia. In this case, along whatever path N' moves to the right, NN', or $\hat{N}$N', the maximum per capita consumption, will be higher than $C_g^* N$. If N' moves along N^*N'R', however, the maximum will be at K_g^*, but then the maximum level of consumption would have been raised from $C_g^* N$ to $C_g^* \hat{N}$. In all cases we see the immutability of the impossibility theorem is not impossible.

III

It is seen from eqn. (4), by multiplying both sides by K, that the maximum per capita consumption is obtained at a point when the volume of savings or investment equals profits. This can be shown even without assuming the rate of growth of capital to be constant. Using the linear homogeneous Cobb-Donglas production function we have,

$$(5) \quad Q = u L^{\alpha} K^{\beta} \qquad \alpha + \beta = 1$$

$$(6) \quad \dot{K} = S u L^{\alpha} K^{\beta} \text{ (S being the rate of savings)}$$

$$(7) \quad L = L_o e^{\lambda t} \qquad \text{(labour grows at the rate } \lambda \text{ per period)}$$

substituting (7) into (6) and solving we get

$$(8) \quad K = \left\{ \frac{S L_o^{\alpha} (1-\alpha)}{\alpha \lambda} e^{\alpha \lambda t} + K_o^{1-\beta} - \frac{S L_o^{\alpha} (1-\beta)}{\alpha \lambda} \right\}^{\frac{1}{1-\beta}}$$

Substituting (8) and (7) in (5), we have

$$(9) \quad Q = u L_o^{\alpha} e^{\alpha \lambda t} \left\{ \frac{S L_o^{\alpha} (1-\beta)}{\alpha \lambda} e^{\alpha \lambda t} + K_o^{1-\beta} - \frac{S L_o^{\alpha} (1-\beta)}{\alpha \lambda} \right\}^{\frac{\beta}{1-\beta}}$$

Therefore,

(10) $C = Q - SQ$

$$C^{*} = (1-S)\, Q^{*} \quad \left(C^{*} = \frac{C}{L}\ ,\ Q^{*} = \frac{Q}{L}\right)$$

(11) $$= (1-S)\, L_o^{-\beta} e^{-\beta\lambda t} \left\{ \frac{SL_o^{\alpha}(1-\beta)}{\alpha\lambda} e^{\alpha\lambda t} + K_o^{1-\beta} - \frac{SL_o^{\alpha}(1-\beta)}{\alpha\lambda} \right\}^{\frac{\beta}{1-\beta}}$$

Differentiating (11) w.r.t. S and equating to zero, we have as the necessary condition for the maximisation of C^{*},

$$\frac{SL_o^{\alpha}}{\lambda} e^{\alpha\lambda t} + K_o^{1-\beta} - \frac{SL_o^{\alpha}}{\lambda} = (1-S)\frac{\beta}{1-\beta}\left[\frac{L_o^{\alpha} e^{\alpha\lambda t}}{\lambda} - \frac{L_o^{\alpha}}{\lambda}\right]$$

Dividing through by $e^{\alpha\lambda t}$ and letting $t \to \infty$

$$\therefore \frac{SL_o^{\alpha}}{\lambda} = (1-S)\frac{\beta}{1-\beta}\cdot\frac{L_o^{\alpha}}{\lambda}$$

(12) $S = \beta$

This is substantially the same result as arrived at earlier. It states that if the productive activity is characterised by a production function of the linear Cobb-Douglas type, then the per capita consumption will be maximised asymptotically if the rate of saving is equal to the share of capital. If β, the share of capital in the linear Cobb-Douglas, lies between .2 and .3, then it may not be a mere coincidence that the rates of savings have tended to settle between these limits in most of the developed countries.

However, if we can rely on the validity of the Cobb-Douglas production function, it can be safely stated that in the underdeveloped countries where the rates of savings are around 10% of the national income, the stage of capital accumulation when the per capita consumption will have been maximised is yet very far off indead. Hence the discussion of the boundedness of the per capita consumption stream through time in the context of development of the traditional economies is of academic nature only. For so long as K^{*} does not attain the value which maximises the per capita consumption in the absence of technical progress, i.e., K_g^*, any increase in the value of K^* through capital accumulation will result in an increase in C^*, i.e., the per capita consumption.

IV

It follows from the above (fig.1) that so long as $K^* < K_g^*$, a reduction in the rate of growth of labour or population below η will move K^* towards K_g^* and so the per capita consumption will increase. But if $K^* > K_g^*$, a decrease in the rate of growth of population below η will move K^* even further away from K_g^* and C^* will decrease. In the latter case C^* will increase only when the rate of growth of population increases above η. As it is obvious that in most of the under-developed countries, K^* is far below K_g^*, any reduction in the rate of growth of population will lead to increase in the per capita consumption over what would have been attained in the absence of such a decrease of population growth. However, changes in the rate of growth of population will not affect the maximum level of per capita consumption that is attainable in the absence of technical progress.

When technical progress in introduced the 'impossibility theorem' may cease to hold, if the production function is altered in such a way that the second inequality in (1) ceases to hold. In that case the per capita consumption will rise without any limit depending upon the intensity of the technical progress. It should be noted, however, that technical progress is largely dependent upon the rate of capital accumulation.[1] So even if credence is given to the 'impossibility theorem' the role of capital accumulation from domestic savings or foreign borrowings cannot be minimised.

A. Qayum

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1) K.J. Arrow, "The Economic Implications of Learning by Doing", The Review of Economic Studies, June 1962.

See also Zvi Griliches, "The Sources of Measured Productivity Growth: United States Agriculture, 1940-60", Journal of Political Economy, Aug. 1963.