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On The Transportation Problem with Mixed Constraints

By

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1. Introduction:

The standard transportation problem in linear programming is well known and may be stated, mathematically, as follows:

minimize $\sum_i \sum_j c_{ij} \; x_{ij}$,

subject to $\sum_j x_{ij} = a_i \qquad i = 1, 2, \ldots, m,$

$\sum_i x_{ij} = b_j \qquad j = 1, 2, \ldots, n,$

$x_{ij} \geq 0,\; a_i \geq 0,\; b_i \geq 0,$ for all i and j,

$\sum_i a_i = \sum_j b_j,$

where $a_1, \ldots, a_m$ are the supplies available at the m sources and $b_1, \ldots, b_n$ are the demands at the n destinations. The first statement of this type of problem is usually ascribed to Hitchcock 6. Dantzig[4] gave the standard form and applied the simplex method to this special linear programming problem. An important limitation to the standard transportation problem is that the availabilities at the sources and the requirements at the destinations are fixed quantities and met exactly by the solution. The only exception to this is when there is unbalance between the total available and the total requi ed, in this case one introduces a dummy source or a dummy destination with zero costs. In real situations, it is frequently desirable to be more flexible and specify minimum quantitites or maximum quantitites which may be taken from/or received at some sources destinations, whilst specifying the exact amount, to be taken from/

received at others. Variants of the standard transportation model in which the origin and/or destination constraints are inequations as opposed to the usual equations have been considered by Appa[1]. He considered 81 unimodular problems, however, he did not deal with the case where origin and destination constraints are of mixel type. Brigden[3] and klingman et al[9] have extended some of Appa's ideas to include this general case of mixed type. It has been shown by klingmen et al[9] that the mixed transportation model is equivalent to a standard transportation problem having only one additional origin and destination. No proof of the optimality of the obtained solution for the mixed model has been given in their reference (9). The purpose of this paper is to give this optimality proof. In section 2, we prove in theorem 1 the optimality of the mixed-solution obtained from the optimal solution to the equivalent standard problem. In section 3, we present a method for finding alternative optimal basic solutions (if there are more than one) to the mixed transportation problem. We give little computational experience in section 5. In the appendix, we present a computer package program for solving the mixed model.

2. The Mixed Transportation Problem and the Related Standard Transportation Problem.

The transportation problem with mixed constraints, called the mixed problem (MP)[3] has been defined as:

Assume that the origin index set $I = \{1, 2, \ldots, m\}$ is partitioned into sets I_1, I_2, I_3, where the origin O_i, $i \in I_1$ must distribute at least a_i units of goods, O_i, $i \in I_2$ must distribute exactly a_i units, and O_i, $i \in I_3$ may distribute at most a_i units of goods. Also suppose that the distrination index set $J = \{1, 2, \ldots, n\}$ is partitioned into sets J_1, J_2, J_3 where destination D_j, $j \in J_1$ must receive at least b_j units of supply, D_j, $j \in J_2$ must receive exactly b_j units and D_j, $j \in J_3$ receives at most b_j units of supply. The objective is to minimize the total shipping cost.

In mathematical form MP has the form:

MP: minimize $z = \sum_{i \in I} \sum_{j \in J} C_{ij} \; X_{ij}$,

subject to

$$\sum_{j \in J} X_{ij} \geq a_i, \quad i \in I_1,$$

$$\sum_{j \in J} X_{ij} = a_i, \quad i \in I_2,$$

$$\sum_{j \in J} X_{ij} \leq a_i, \quad i \in I_3,$$

$$\sum_{i \in I} X_{ij} \geq b_j, \quad j \in J_1,$$

$$\sum_{i \in I} X_{ij} = b_j, \quad j \in J_2,$$

$$\sum_{i \in I} X_{ij} \leq b_j, \quad j \in J_3,$$

$$X_{ij} \geq 0, \text{ for all } i \in I \text{ and } j \in J,$$

where x_{ij} and c_{ij} denote, respectively, the amount shipped and the cost of shipping from o_i to D_j. It is assumed that $C_{ij} > 0$ for all $(i,j) \in I_1 \times J_1$; this condition guarantees that z is bounded below in the feasible region[3]. If $I_1 = \phi$ or $J_1 = \phi$, where ϕ is the null set, MP may have no feasible solution. For example, it is easy to see that if $I_1 = \phi$ and $\sum_{j \in J_2} b_j + \sum_{j \in J_1} b_j \geqslant \sum_{i \in I} a_i$, then MP has no feasible solution.

The following are the necessary and sufficient conditions for the existence of a feasible solution to MP for some special cases:

(a) if $I_1 = \phi$ it is required to have

$$\sum_{j \in J_2} b_j + \sum_{j \in J_1} b_j \leqslant \sum_{i=1}^{m} a_i ;$$

(b) if $J_1 = \phi$, then it is required that

$$\sum_{i \in I_2} a_i + \sum_{i \in I_1} a_i \leqslant \sum_{j=1}^{n} b_j ;$$

and

(c) if $I_1 = \phi$ and $J_1 = \phi$, then $\sum_{j \in J_2} b_j \leqslant \sum_{i \in I} a_i$ and

$\sum_{i \in I_2} a_i \leqslant \sum_{j \in J} b_j$ must hold.

The mixed transportation model can be used to investigate the effects of changing the supply at different origins and/or changing the demand at various destinations. For example, in a transportation problem with given demands and some supplies at fixed levels, the cost of reducing supply at certain unfixed origins and increasing supply at others could be determined, and thus the optimum re-location of supply at various origins are acheived. Similarly the model can be used to investigate cost reductions resulting from simultaneously increasing or/and decreasing supply and demands.

The mixed model could also be used to incorporate priorities in allocation of funds to various agencies which fund various projects. Assume that certain agencies are to be budgeted an amount of money at least equal to a_i, other agencies will have an exact budget of a_i, and some low priority agencies will receive at most a_i. At the same time, the projects which are to be funded have priorities which have demands of at least b_j for high priority projects, exactly b_j for some projects and at most b_j for low priority projects. If c_{ij} is the per unit cost of agency i processing funds to project j, then the mixed transportation model may be used to find a minimum cost allocation of funds to the various agencies and projects subject to certain priorities.

Models related to the mixed transportation problem include, the purchase storage problem[8], production scheduling problem[2], and the carter problem[7]. Klingman et al[9] have shown that the MP may be solved by transforming it to an equivalent standard transportation broblem. The proposed related standard transportation problem (RTP) has the form:

RTP: minimize $f = \sum_{i \in I'} \sum_{j \in J'} c_{ij} \; x_{ij}$,

subject to

$$\sum_{j \in J'} x_{ij} = a_i \; , \qquad i \in I, \tag{1}$$

$$\sum_{j \in J'} x_{m+1,j} = N - \sum_{i \in I} a_i = a_{m+1},$$

$$\sum_{i \in I'} x_{ij} = b_j \qquad , \quad j \in J \quad ,$$

$$\sum_{i \in I'} x_{i,n+1} = N - \sum_{j \in J} b_j = b_{n+1}, \tag{2}$$

$$x_{ij} \geq 0, \text{ for all } i \in I', \; j \in J',$$

where $I' = \{1, 2, \ldots, m+1\}$, $J' = \{1, 2, \ldots, n+1\}$,

$$c_{i,n+1} = \min_{j \in J_1} c_{ij} \qquad \text{for } i \in I_1 \cup I_2$$

$$c_{i,n+1} = 0 \qquad \text{for } i \in I_3, \tag{3}$$

$$c_{m+1,j} = \min_{i \in I_1} c_{ij} \qquad \text{for } j \in J_1 \cup J_2, \tag{4}$$

$$c_{m+1,j} = 0 \qquad \text{for } j \in J_3, \text{ and } c_{m+1,n+1} = 0 \; .$$

N is a real number which represents an upper bound to the total number of goods shipped, e.g, N could be $> 2 \sum_{j \in J} b_j$.

We prove the following theorem:

<u>Theorem 1:</u>

Let $\hat{x}_{ij}$, $(i,j) \in I^{`} X J^{`}$ be an optimal solution to RTP.

Then $x^*_{ip} = \hat{x}_{ip} + \hat{x}_{i,n+1}$, $i \in I_1 \cup I_2$ and p is a particular column satisfying $C_{ip} = \min_{j \in J_1} C_{ij}$,

$x^*_{rj} = \hat{x}_{rj} + \hat{x}_{m+1,j}$, $j \in J_1 \cup J_2$ and r is a particular row satisfying $c_{rj} = \min_{i \in I_1} C_{ij}$,

$x^*_{ij} = \hat{x}_{ij}$, for all others $(i, j) \in I X J$,

is an optimal solution to MP.

<u>Proof:</u>

We first prove that $\{x^*_{ij}\}$ is a feasible solution to MP.

(i) For $i \in I_1$, we have

$$\sum_{j \in J} X^*_{ij} = \sum_{\substack{j \in J \\ j \neq p}} \hat{x}_{ij} + \hat{x}_{ip} + \hat{x}_{i,n+1} = \sum_{j \in J^{`}} \hat{x}_{ij}.$$

Since $\{\hat{x}_{ij}\}$ is a feasible solution to RTP, then from equation (1) , we get.

$$\sum_{j \in J} X^*_{ij} = a_i.$$

For the particular row r, we have

$$\sum_{j \in J} x^*_{rj} = \sum_{\substack{j \in J \\ j \neq P}} \hat{x}_{rj} + \hat{x}_{rp} + \hat{x}_{r,n+1} + \hat{x}_{m+1,j} = \sum_{j \in J^{`}} \hat{x}_{rj} + \hat{x}_{m+1,j}$$

Since $\hat{x}_{m+1,j} \geqslant 0$ for any $j \in J$, then

$$\sum_{j \in J} x^*_{rj} = a_r + \hat{x}_{m+1,j} \geqslant a_r .$$

Hence, generally $\sum_{j \in J} x^*_{ij} \geqslant a_i \quad , \quad i \in I_1 .$

(ii) For $i \in I_2$, we have

$$\sum_{j \in J} x^*_{ij} = \sum_{\substack{j \in J \\ j \neq p}} \hat{x}_{ij} + \hat{x}_{ip} + \hat{x}_{i,n+1} = \sum_{j \in J^\frown} \hat{x}_{ij} = a_i .$$

(iii) For $i \in I_3$,

$$\sum_{j \in J} x^*_{ij} = \sum_{j \in J} \hat{x}_{ij} \leqslant \sum_{j \in J} \hat{x}_{ij} + \hat{x}_{i,n+1} = \sum_{j \in J^\frown} \hat{x}_{ij} = a_i .$$

A similar argument shows that the column constraints of the MP are satisfied. Obviously, since $\hat{x}_{ij} \geqslant 0$ for all $(i,j) \in I^\frown \times J^\frown$, then $x^*_{ij} \geqslant 0$ for all $(i,j) \in I \times J$. This completes the proof of feasibility. Now we prove the optimality by the duality theory of linear programming.

Let the dual problem of MP be:

DMP: Maximize $\sum_{i=1}^{m} a_i s_i + \sum_{j=1}^{n} b_j t_j$,

subject to $s_i + t_j \leqslant C_{ij}$, $i = 1, \ldots, m$ and $j = 1, \ldots, n$,

$s_i \geqslant 0$ for $i \in I_1$,

$s_i \leqslant 0$ for $i \in I_3$,

$t_j \geqslant 0$, for $j \in J_1$, and $t_j \leqslant 0$ for $j \in J_3$,

and the dual problem of RTP be:

$$\text{DRTP: maximize } \sum_{i=1}^{m+1} a_i u_i + \sum_{j=1}^{n+1} b_j v_j,$$

$$\text{Subject to } \; U_i + V_j \leqslant C_{ij}, \quad i=1, \ldots, m+1 \text{ and } j=1, \ldots, n+1. \quad \ldots(5)$$

Firstly, we show that DRTP has an optimal solution $\{(\hat{u}_i, \hat{v}_j)\}$ with

$\hat{U}_{m+1} = \hat{V}_{n+1} = 0$.

It is well known that the optimal dual solution to any transportation problem is not unique and then we may set $\hat{u}_{m+1} = 0$.

From equation (2), we have

$$\sum_{i \in I} \hat{x}_{i,n+1} + \hat{x}_{m+1,n+1} = N - \sum_{j \in J} b_j > \sum_{j \in J} b_j,$$

that is, $\hat{x}_{m+1,n+1} > \sum_{j \in J} b_j - \sum_{i \in I} \hat{x}_{i,n+1}$.

Since the total flow from origin 1 to m through destination n+1 in any optimal solution to RTP need not exceed $\sum_{j \in J} b_j$, then we must have $\hat{x}_{m+1,n+1} > 0$, i.e, $\hat{x}_{m+1,n+1}$ is a basic variable in the final solution.

Hence, $\hat{U}_{m+1} + \hat{V}_{n+1} = C_{m+1,n+1} = 0$,

which implies that $V_{n+1} = 0$.

Secondly, we show that

$$s_i = \hat{U}_i \quad , \quad i = 1, \ldots, m,$$

$$t_j = \hat{V}_j \quad , \quad j = 1, \ldots, n,$$

is a feasible solution to DMP.

From (5) we get

$\hat{u}_i + \hat{v}_j \leqslant C_{ij}$, for i = 1, 2, ..., m, & j = 1, 2, ..., n.

For $i \in I_3$,

$\hat{\ddot{U}}_i + \hat{v}_{n+1} \leqslant C_{i,n+1} = 0$ (from (3)).

Since $\hat{v}_{n+1} = 0$, then $\hat{U}_i \leqslant 0$.

For $i \in I_1$;

if $\hat{x}_{i,n+1} > 0$, then $\hat{U}_i + \hat{V}_{n+1} = C_{i,n+1} \geqslant 0$,

Hence $\hat{U}_i \geqslant 0$.

If $\hat{x}_{i,n+1}$ is not basic, then $\hat{x}_{iq} > 0$ for at least a column $q \in J$ (a property of a basic feasible transportation solution).

Therefore, $\hat{U}_i + \hat{V}_q = C_{iq}$.

From (4), we have $\hat{U}_{m+1} + \hat{V}_q \leqslant C_{m+1,q} \leqslant C_{iq}$.

Hence, $\hat{U}_{m+1} + \hat{V}_q \leqslant \hat{U}_i + \hat{V}_q$ which implies that

$$\hat{u}_i \geqslant 0.$$

In a similar way it can be shown that

$\hat{V}_j \geqslant 0$, for $j \in J_1$ and $\hat{V}_j \leqslant 0$ fro $j \in J_3$.

Thus $\hat{U}_i$, i=1, ..., m and $\hat{V}_j$, j=1,...,n is a feasible solution to DMP. Now, from the duality theory, the primal and dual objective values of the RTP are equals, i.e,

$$\hat{f} = \sum_{i \in I'} \sum_{j \in J'} c_{ij} \hat{x}_{ij} = \sum_{i=1}^{m+1} a_i \hat{U}_i + \sum_{j=1}^{n+1} b_j \hat{V}_j .$$

Due to the standard procedure of the least-cost paths (wagner[10], P.172) we have.

$$\hat{f} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} X^*_{ij} . \qquad (6)$$

From (6) and the feasibility of $\{(\hat{u}_i, \hat{v}_j)\}$ to DMP, we get

$$\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x^*_{ij} = \sum_{i=1}^{m} a_i \hat{U}_i + \sum_{j=1}^{n} b_j \hat{V}_j .$$

Since the feasible solutions $\{X^*_{ij}\}$ and $\{(\hat{U}_i, \hat{V}_j)\}$ to the primal and dual, respectively, of the MP give equality in their objective functions, then $\{X^*_{ij}\}$ is an optimal solution to MP.

3. Alternative Optimal Solutions to the Mixed Transportation Problem

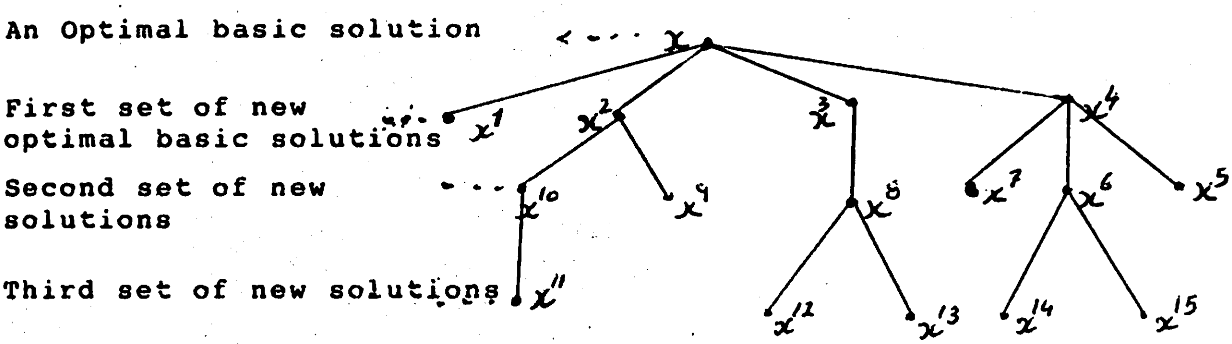
From theorem 1 we see that an optimal solution to the mixed model can be obtained from the optimal solution to the related standard transportation problem. Consequently, if the related problem has alternative optima, then the mixed problem has alternative optima as well. Sometimes, useful information can be obtained from the knowledge of different optimal solutions. Hence, it is desirable to show how alternative optima of the MP can be found.

If the optimal basic solution to RTP is not degenerate and if $u_i + V_j < C_{ij}$ for each nonbasic cell (i, j) in the optimal transportation tableau (the notation used here is that which is standard for transportation problems), then the optimal basic solution obtained is unique and consequently MP has a unique optimal solution. However, if $u_i + V_j = C_{ij}$ for one (or more) nonbasic cell(s) (i, j), then the associated nonbasic variable(s) X_{ij} can be inserted into the basic set to yield a different optimal solution. This is true since the nonbasic variable x_{ij} associated with the reduced cost coefficient $u_i + v_j - C_{ij}=0$ enters the basic set without increasing the value of the objective function. This fact suggests a procedure for finding alternative optima for RTP.

Starting with an optimal basic solution X to RTP having zero reduced cost coefficients, we generate a new set of different optimal solutions where each new solution differes from X only in that one variable in the basic set is changed, i.e, that variable which had $u_i + V_j - C_{ij}=0$. Each such solution is called adjacent to X. We repeat the same process with each of the new optimal basic solution, that is generating all optimal basic solutions adjacent to each optimal basic solution found. We continue with each set of new optimal solutions until it is no longer possible to find any optimal solutions different from those already obtained in the previous steps. Since the number of optimal basic solutions is finite, then the procedure will come to an end.

In schematic form, the process may have a tree-like structure as shown for a hypothetical example in figure 1.

Figure 1



In this example, there are fifteen different optimal basic solutions.

After generating a sufficient number of optimal basic solutions, we only repeat solutions already obtained in previous steps. To prevent the computation of optimal solutions which have already been generated, it is necessary to keep record for all optimal basic solutions that previously and currently determined. The "book-keeping" requires a list of the indices of the basic variables for each optimal basic solution held. Since, the same optimal solution may be generated from more than one optimal solution, then a comparison between the currently determined one and the previously generated solutions, which are recorded in the book-keeping, is an essential process to avoid considering redundant optimal solutions. This comparison process will be repeated often during the generation of theoptimal solutions,so careful attention must be paid to the coding of the comparison routine.

4. Numerical Example

Let us consider the following 3X4 transportation problem with mixed constraints.

Minimize:

$$Z = x_{11} + 6x_{12} + 2x_{13} + 5x_{14} + 7x_{21} + 3x_{22} + x_{23} + 6x_{24} + 9x_{31}$$
$$4x_{32} + 5x_{33} + 4x_{34},$$

Subject to:

$$\begin{array}{llll} x_{11} + x_{12} + x_{13} + x_{14} & & & = 20 \\ & x_{21} + x_{22} + x_{23} + x_{24} & & \geq 16 \\ & & x_{31} + x_{32} + x_{33} + x_{34} & \leq 25 \\ x_{11} & + x_{21} & + x_{31} & \geq 11 \\ x_{12} & + x_{22} & + x_{32} & \leq 13 \\ x_{13} & + x_{23} & + x_{33} & \geq 17 \\ x_{14} & + x_{24} & + x_{34} & = 14, \end{array}$$

$$x_{ij} \geq 0 \quad \text{for all i and j.}$$

Choosing $N = 165$ $(>2 \sum_{j \in J} b_j)$, the 4X5 RTP takes the form:

minimize $f = Z + X_{15} + X_{25} + 7X_{41} + X_{43} + 6X_{44}$.

Subject to:

$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 20$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 16$$

$$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} = 25$$

$$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} = 104$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 11$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 13$$

$$x_{13} + x_{23} + x_{33} + x_{43} = 17$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 14$$

$$x_{15} + x_{25} + x_{35} + x_{45} = 110,$$

$$x_{ij} \geqslant 0 \quad \text{for all } i \text{ and } j,$$

Solving the RTP by the U-V transportation method[5], we get the optimal transportation tableau:

Tableau 1

O_i \ D_j	D_1	D_2	D_3	D_4	D_5	Total
O_1	1) 11	6)	2)	5)	1) 9	20
O_2	7)	4)	1) 16	6)	1)	16
O_3	9)	4)	5)	4) 14	0) 11	25
O_4	7)	0) 13	1) 1	6)	0) 90	104
Total	11	13	17	14	110	165

The minimum Cost f=93.

The nonbasic variables x_{13} and x_{14} can be inserted into the basic set to yield two different optimal basic solutions. These alternative solutions are represented by the two tableau X:

Tableau 2

O_i \ D_j	D_1	D_2	D_3	D_4	D_5	Total
O_1	1 / 11	6	2	5 / 9	1	20
O_2	7	3	1 / 16	6	1	16
O_3	9	4	5	4 / 5	0 / 20	25
O_4	7	0 / 13	1 / 1	6	0 / 90	104
Total	11	13	17	14	110	165

Tableau 3

O_i \ D_j	D_1	D_2	D_3	D_4	D_5	Total
O_1	1 / 11	6	2 / 1	5	1 / 8	20
O_2	7	3	1 / 16	6	1	16
O_3	9	4	5	4 / 14	0 / 11	25
O_4	7	0 / 13	1	6	0 / 91	104
Total	11	13	17	14	110	165

From Tableau 2, we find that nonbasic variable x_{13} and the basic variable x_{43} can be interchanged to get the following optimal tableau:

Tableau 4

O_i \ O_j	D_1	D_2	D_3	D_4	D_5	Total
O_1	1 / 11	6	2 / 1	5 / 8	1	20
O_2	7	3	1 / 16	6	1	16
O_3	9	4	5	4	0 / 19	25
O_4	7	0 / 13	1	6	0 / 91	104
Total	11	13	17	14	110	165

No new optimal basic solutions can be generated from Tableau 3 or Tableau 4. Hence, the 4x5 RTP has four alternative optimal basic solutions. Constructing the alternative optimal solutions to the original mixed problem in accordance with the relations stated in theorem 1, we get:

From Tableau 1: $x_{11} = 11+9 = 20$, $x_{23} = 16+1 = 17$, $x_{34} = 14$;

from Tableau 2: $x_{11} = 11$, $x_{14} = 9$, $x_{23} = 16+1 = 17$, $x_{34} = 5$;

from Tableau 3: $x_{11} = 11+8 = 19$, $x_{13} = 1$, $x_{23} = 16$, $x_{34} = 14$;

from Tableau 4: $x_{11} = 11$, $x_{13} = 1$, $x_{23} = 16$, $x_{34} = 6$, $x_{14} = 8$.

For each optimal solution generated, the associated minimum cost is equal to 93. Although, the minimum cost is the same for any of the alternative optima of the mixed problem, there may exist a certain optimal solution which is preferable by the decision maker for reasons others than the criterion cost. For example, the optimal solution obtained from Tableau 1 may be preferable because each origin supplies only one destination with the reqiured quantity.

5- Computational Experience.

The method, described in section 2 and 3, for obtaining alternative optimal solutions to the mixed transportation problem has been programmed in FORTRAN. The program has been tested on a number of manufactured problems on PERKIN-ELMER 3220 computer at the computing department, Institute of National Planning. During the computation we need to store a transportation tableau and an integer array of dimension (m+n+1) Xu for the list of the basic variable indices of the optimal solutions, where u is an upper bound for the number of optimal basic solutions. Since the theoretical estimate of u may be larger than the capacity of the core storage of the computer and to avoid use of an array with dynamic bound, we use the value of u which fits the set of basic indices into the core memory while including in the program a device indicating that incorrect estimate of u has been used. A main feature of the proecedure computation is that the only arithmetic operations used during the process are the addition and subtraction on integers numbers, therefore, no problem of controlling the machine round-off error is encounteredy; inaddition, the use of fixed point operations operations on a computer is less costly in time than the floating points operations. A largest problem processed was of six origins and ten destinations for which 96 optimal basic solutions were prodused in 10 minutes. A testing degenerate problem having 3 origins and 6 destinations gave 22 alternative optima, one of which has been printed 10 times (in the case of degeneracy, the same extreme point may be represented by different basic solutions). In the appendix, the computer results of the numerical example shown in section 3 are presented.

ALTERNATIVE OPTIMAL SOLUTION OF RELATED TRANSP... PROPLEM

X11= 11   X15= 9   X23= 16   X34= 14   X35= 11   X42= 13   X43= 1   X45= 90

X11= 11   X14= 9   X23= 16   X34= 5   X35= 20   X42= 13   X43= 1   X45= 90

X11= 11   X13= 1   X15= 8   X23= 16   X34= 14   X35= 11   X42= 13   X45= 91

X11= 11   X13= 1   X14= 8   X23= 16   X34= 6   X35= 19   X42= 13   X45= 91

ALTERNATIVE OPTIMAL SOLUTIONS TO THE MIXED TRANSPORTATIION PROPLEM

X11= 20   X23= 17   X34= 14

X11= 11   X14= 9   X23= 17   X34= 5

X11= 19   X13= 1   X23= 16   X34= 14

X11= 11   X13= 1   X23= 16   X34= 6   X14= 8

THE VALUE OF MIN. COST= 93

ND OF PROGRAM

```
C     ------------------------------------------------------------------
C
C      APROGRAM FOR FINDING ALL OPTIMAL SOLUTIONS OF THE
C      TRANSPORTATION PROBLEM WITH MIXED CONSTRAINTS
C
C     ------------------------------------------------------------------
C
      DIMENSION ICOST(7,10),JBAS(7,10),ND(2),NB(16),NS(2),
     -          INB(40,16),JAR(16)
C
C     READ NUMBER OF ORIGINS,NUMBER OF DESTINATIONS
C
      READ(1,1) NORIG,NDEST
      M=NORIG+NDEST
      K1=10000
      ICHANG=0
C     READ NUMBER OF ROWS AND COLUMNS OF DIFFERENT TYPES
      READ(1,11)I1,I2,J1,J2,JPCOL,IROW
C     READ MAX NUMBER
C
      READ(1,9)MNUM
      N2=MNUM+1
      M1=M-1
      M=2*M
      N1=1
      DO 80 I=1,NORIG
   80 READ(1,11)(ICOST(I,J),J=1,NDEST)
      DO 10 I=1,NORIG
   10 READ(1,11)(JBAS(I,J),J=1,NDEST)
    1 FORMAT(2I4)
    9 FORMAT(I4)
   11 FORMAT(10I5)
  520 DO 8 IR=1,M1
    8 INB(N1,IR)=0
COMMENT
C------
C     DETERMINATION OF THE INDICATOR OF THE INITIAL OPTIMAL
C     SOLUTION
      L1=1
  540 NROW=1
  550 NCOL=1
  560 IF(JBAS(NROW,NCOL).LT.K1)GO TO 570
      ND(1)=NROW
      ND(2)=NCOL
      CALL PACK(2,ND,NPK)
      INB(N1,L1)=NPK
      L1=L1+1
  570 NCOL=NCOL+1
      IF(NCOL.LE.NDEST)GO TO 560
  590 NROW=NROW+1
      IF(NROW.LE.NORIG) GO TO 550
COMMENT
C------
C     WRITE THE INDICATOR OF THE FIRST OPTIMAL SOLUTION
C
      CALL WRITE(N1,M1,K1,JBAS,INB)
COMMENT
C------
C     DETERMINATION OF NEIGHBOUR INDICATOR(S) FOR THE CURRENT ONE
C
  700 IEND=0
      ICONT=0
  740 NROW=1
  750 NCOL=1
```

```
  760 IF(JBAS(NROW,NCOL).EQ.0) GO TO 720
  770 NCOL=NCOL+1
      IF(NCOL.LE.NDEST)GO TO 760
      NROW=NROW+1
      IF(NROW.LE.NORIG)GO TO 750
      GO TO 710
  720 NROW1=NROW
      NCOL1=NCOL
      ND(1)=NROW
      ND(2)=NCOL
      CALL PACK(2,ND,NPK1)
      CALL LOOP(NORIG,NDEST,JBAS,NROW1,NCOL1,ICONT,NROW2,NCOL2,K1)
      ND(1)=NROW2
      ND(2)=NCOL2
      CALL PACK(2,ND,NPK2)
      DO 20 I1=1,M1
      IF (INB(N1,I1).NE.NPK2)GO TO 30
      NB(I1)=NPK1
      GO TO 20
   30 NB(I1)=INB(N1,I1)
   20 CONTINUE
COMMENT
C------
C     A NEIGHBOUR INDICATOR IS NOW ESTABLISHED IN THE ARRAY NB
C     WE CHECK IF THAT INDICATOR IS IN S OR IN W
C
      CALL TEST(N1,N2,M1,INB,NB,IFLAG,NFLAG,MNUM)
      IF (IFLAG.EQ.1)GO TO 1400
      IF(NFLAG.EQ.1)GO TO 1600
COMMENT
C------
C     UPDATE THE POINTER N2 AND STORE THE NEIGHBOURING INDICATOR
C
      N2=N2-1
      DO 40 IL=1,M1
   40 INB(N2,IL)=NB(IL)
 1400 CONTINUE
 1600 CONTINUE
COMMENT
C------
C     TO CONSIDER ANOTHER NEIGHBOUR INDICATOR,IF THERE IS ANY,
C     GO TO 770
C
      GO TO 770
  710 CONTINUE
      N11=N1+1
COMMENT
C------
C     CONSIDER THE LAST STORED INDICATOR IN S
C
      DO 70 JV=1,M1
      NB(JV)=0
   70 NB(JV)=INB(N2,JV)
      DO 501 IW=1,M1
  501 JAR(IW)=0
COMMENT
C------
C     DETERMINATION OF THE DIFFERENT COMPONENTS BETWEEN THE LAST
C     CALCULATED INDICATOR AND THE CONSIDERING ONE,AND UPDATE
C     THE BASIC ELEMENTS OF THE OPTIMAL SOLUTION
C
      DO 502 IZ=1,M1
      IP=NB(IZ)
      DO 50 IX=1,M1
      IF (JAR(IX).EQ.1)GO TO 50
      IF (INB(N1,IX).EQ.IP)GO TO 505
```

```
      GO TO 50
  505 JAR(IX)=1
      GO TO 502
   50 CONTINUE
      CALL UNPACK(M1,IP,NS)
      NROW1=NS(1)
      NCOL1=NS(2)
      ICONT=1
      CALL LOOP(NORIG,NDEST,JBAS,NROW1,NCOL1,ICONT,NROW2,NCOL2,K1)
      IEND=1
  502 CONTINUE
COMMENT
C------
C     IEND=0 INDICATES THAT THE SET W=0,THAT IS ALL OPTIMAL
C     SOLUTIONS ARE OBTAINED. WE CATCH THE END OF THE PROGRAM
C     BY GO TO 75
C
      IF(IEND.EQ.0)GO TO 75
COMMENT
C------
C     UPDATE THE TRANSPORTATION TABLEAU
C
      CALL UVMETHOD(NORIG,NDEST,ICOST,M1,JBAS,K1)
      N1=N1+1
      IF(N1.EQ.N2)GO TO 73
      DO 2000 IB=1,M1
 2000 INB(N1,IB)=INB(N2,IB)
COMMENT
C------
C     WRITE THE NEW CONSIDERED INDICATOR
C
      CALL WRITE(N1,M1,K1,JBAS,INB)
      N2=N2+1
      IF(N2.GT.MNUM)GO TO 75
COMMENT
C------
C     TO GET THE NEIGHBOUR INDICATOR OF THE CURRENT ONE,GO TO 700
C
      GO TO 700
   73 WRITE(2,77)
   77 FORMAT(3X,'THERE IS AN OVERLAP . ')
   75 CONTINUE
      STOP
      END
      SUBROUTINE WRITE(N1,M1,K1,JBAS,INB)
      DIMENSION JBAS(7,10),INB(40,16),NS(2)
      WRITE(2,19)N1
   19 FORMAT(2X,'OPTIMAL SOLUTION NO. ',I3)
      DO 101 IA=1,M1
      IB=INB(N1,IA)
      CALL UNPACK(M1,IB,NS)
      N=NS(1)
      L=NS(2)
      IBAS=JBAS(N,L)-K1
      WRITE(2,12)IBAS,N,L
   12 FORMAT(2X,' X    =',1X,I4/4X,2I2)
  101 CONTINUE
      RETURN
      END
C
C-----------------------------------------------------------------
C     A SUBROUTINE TO IDENTIFY BASIS LOOP AND UPDATE ITS ENTRIES
C-----------------------------------------------------------------
C
      SUBROUTINE LOOP(NORIG,NDEST,KBAS,NROW1,NCOL1,ICONT,NROW2
     -,NCOL2,K1)
```

```
      DIMENSION KBAS(7,10)
     -         ,INET1(7),INET2(10),INET(34)
      K=NORIG+NDEST
      K=2*K
COMMENT
C------
C         *INITIALIZE
C           .SET K2.GE.(10*K1)
C
      DO 33 I=1,NORIG
   33 INET1(I)=0
      DO 44 J=1,NDEST
   44 INET 2(J)=0
      DO 45 IC=1,K
   45 INET(IC)=0
      K2=8*K1
      L2=1
COMMENT
C------
C        *START SEARCHING FOR THE BASIS LOOP
C        ..........................................
C        **FROM HERE UNTIL JUST BEFORE 900 OUR MAIN TASKS ARE
C           1.RECOGNIZING EACH ELEMENT OF LOOP AND BUTTING ITS ROW
C            AND COLUMN NUMBERS IN TWO ADJACENT LOCATIONS OF ARRAY
C            INET
C          .2.DEYECT ANY UNEXPECTED ERRORS
C        ...................................................................
      I=1
      INET(I)=NROW1
      I1=I+1
      INET(I1)=NCOL1
      NROW=NROW1
      NCOL=1
      I=I+2
  100 IF(KBAS(NROW,NCOL).LT.K1) GO TO 160
  120 IF(NCOL.EQ.NCOL1)GO TO 160
  140 INET(I)=NROW
      I1=I+1
      INET(I1)=NCOL
      I=I+2
      GO TO 220
  160 NCOL=NCOL+1
      IF(NCOL.LT.NDEST)GO TO 100
C
C          *NO BASIS ELEMENT IN THE ROW OF THE ENTERING.
C            ERROR TYPE 1
C
  200 WRITE(2,10)
   10 FORMAT(2X,'ERROR1')
      RETURN
  220 INET2(NCOL)=1
      NROW=1
  240 IF(KBAS(NROW,NCOL).GE.K1)GO TO 380
  260 NROW=NROW+1
      IF(NROW.LE.NORIG)GO TO 240
  300 I=I-2
      IF(I.LE.0)GO TO 360
  340 NROW=INET(I)
      I1=I+1
      NCOL=INET(I1)
      INET2(NCOL)=0
      GO TO 4520
C
C     *THERE IS NO BASIS LOOP,ERROR TYPE 2
C
  360 WRITE(2,20)
```

```
   20 FORMAT(2X,'ERROR 2')
      RETURN
  380 I2=I-2
      IF(NROW.EQ.INET(I2))GO TO 260
  400 IF(INET1(NROW).NE.0)GO TO 300
  420 INET(I)=NROW
      I1=I+1
      INET(I1)=NCOL
      I=I+2
      IF(NROW.NE.NROW1)GO TO 480
  460 I=I-2
      GO TO 300
  480 INET1(NROW)=1
      NCOL=1
 4500 IF(KBAS(NROW,NCOL).GE.K1)GO TO 4560
 4520 NCOL=NCOL+1
      IF(NCOL.LE.NDEST)GO TO 4500
      GO TO 640
 4560 I1=I-1
      IF(NCOL.EQ.INET(I1))GO TO 4520
 4580 IF(INET2(NCOL).NE.0)GO TO 640
  600 INET(I)=NROW
      I1=I+1
      INET(I1)=NCOL
      I=I-2
      IF(NCOL.EQ.NCOL1)GO TO 4700
      GO TO 220
  640 I=I-2
      IF(I.LE.0)GO TO 360
  680 NROW=INET(I)
      I1=I+1
      NCOL=INET(I1)
      INET1(NROW)=0
      GO TO 860
 4700 I=3
C
C       *CACULATE THE MINIMUM ENTRY IN BASIS LOOP
C
      ISAV=K2
 4720 NROW=INET(I)
      I1=I+1
      NCOL=INET(I1)
      IF(KBAS(NROW,NCOL).GE.ISAV)GO TO 4780
 4760 ISAV=KBAS(NROW,NCOL)
      NROW2=NROW
      NCOL2=NCOL
 4780 IF(NCOL.EQ.NCOL1)GO TO 860
  800 I=I+4
      IF(I.LE.K)GO TO 4720
C
C
C        *THE BASIS LOOP AS MORE ENTRIES THAN THERE ARE IN THE BASIA.
C        ERROR TYPE 3
C
      WRITE(2,30)
   30 FORMAT(2X,'ERROR 3')
      RETURN
  860 IF(ISAV.LT.K2)GO TO 900
C
C       *THERE IS NO ELEMENT IN THE BASIS LOOP.LT.K2.
C       ERROR TYPE 4
C
      WRITE(2,40)
   40 FORMAT(2X,'ERROR 4')
      RETURN
C     ...........................................................
```

```
C     *NOW THE CATIONS OF ALL ELEMENT IN THE BASIS ARE SEQUENTIALLY
C      ARRANGED(PAIRWISE)IN ARRAY INET AND ISAV CONTAINS THE
C      MINIMUM ENTRY IN THE LOOP,SO WE CAN UPDATE THE BASIC SOLUTION
C      BY SUCCESSIVE ADDITIONS AND SUBTRACTIONS
C      FROM THE ELEMENTS OF THE BASIS LOOP
C     .........................................................
  900 CONTINUE
      IF(ICONT.NE 1)GO TO 1444
      J=-1
      NROW=INET(1)
      NCOL=INET(2)
      KBAS(NROW,NCOL)=ISAV
      ISAV=ISAV-K1
      I=3
  920 NROW=INET(I)
      I1=I+1
      NCOL=INET(I1)
      IF(NROW.NE.NROW2)GO TO 1000
  960 IF(NCOL.NE.NCOL2)GO TO 1000
  980 KBAS(NROW,NCOL)=0
      GO TO 1020
 1000 ISET=KBAS(NROW,NCOL)+J*ISAV
      KBAS(NROW,NCOL)=ISET
 1020 J=-J
      I=I+2
      IF(NCOL.NE.NCOL1)GO TO 920
 1444 CONTINUE
      RETURN
      END
C     ---------------------------------------------------------
C
C     A SUBROUTINE TO CACULATE THE DUAL VARIABLES AND TO UPDATE
C     THE TRANSPORTATION TABLEUX.
C     ---------------------------------------------------------
C
      SUBROUTINE UVMETHOD(NORIG,NDEST,ICOST,M1,JBAS,K1)
      DIMENSION ICOST(7,10),JBAS(7,10),ND(2),
     -IU(7),IV(10),IU1(7),IV1(10)
      DATA IU/7*0/,IV/10*0/,IU1/7*0/,IV1/10*0/
      L1=1
      IU1(1)=1
      NROW=1
      NCOL=1
 3360 IF(JBAS(NROW,NCOL).LT K1)GO TO 3380
 3370 IV(NCOL)=ICOST(NROW,NCOL)-IU(NROW)
      IV1(NCOL)=1
 3380 NCOL=NCOL+1
      IF(NCOL.LE.NDEST)GO TO 3360
 3400 NROW=NROW+1
      IF(NROW.GT.NORIG)GO TO 3500
 3420 IF(IU1(NROW).EQ.1)GO TO 3400
 3430 NCOL=1
 3440 IF(JBAS(NROW,NCOL).GE.K1)GO TO 3470
 3450 NCOL=NCOL+1
      IF(NCOL.GT.NDEST)GO TO 3400
      GO TO 3440
 3470 IF(IV1(NCOL).EQ.0)GO TO 3450
 3480 IU(NROW)=ICOST(NROW,NCOL)-IV(NCOL)
      IU1(NROW)=1
      NCOL=1
      GO TO 3360
 3500 NROW=1
 3510 IF(IU1(NROW).EQ.0)GO TO 3430
 3520 NROW=NROW+1
      IF(NROW.LE.NORIG)GO TO 3510
 3540 NROW=1
```

```
 3550 NCOL=1
 3560 IF(JBAS(NROW,NCOL).LT.K1)GO TO 3620
 3570 NCOL=NCOL+1
      IF(NCOL.LE.NDEST)GO TO 3560
 3590 NROW=NROW+1
      IF(NROW.LE.NORIG)GO TO 3550
      GO TO 3640
 3620 JBAS(NROW,NCOL)=IU(NROW)+IV(NCOL)-ICOST(NROW,NCOL)
      GO TO 3570
 3640 CONTINUE
      RETURN
      END
C
C
      SUBROUTINE PACK(N,NUM,IPACK)
      DIMENSION NUM(N)
      IPACK=0
      DO 10 I=1,N
      IPACK=100*IPACK+NUM(I)
   10 CONTINUE
      RETURN
      END
C
C
      SUBROUTINE UNPACK(M1,IP,NS)
      DIMENSION NS(2)
      NS(1)=IP/100
      NS(2)=MOD(IP,100)
      RETURN
      END
C
C     ----------------------------------------------------------------
C     A SUBROUTINE TO CHECK IF THE CURRENT NEIGHBOURING INDICATOR
C     IS CONSIDERED BEFORE IN THE SERVED OR WAITING SETS
C     ----------------------------------------------------------------
C
      SUBROUTINE TEST(N1,N2,M1,INB,NB,IFLAG,NFLAG,MNUM)
      DIMENSION INB(40,16),NB(16),INET(16)
      IFLAG=0
      NFLAG=0
 1201 J=1
 1202 IF(J.GT.N1.AND.J.LT.N2)GO TO 1200
      DO 2050 I=1,M1
 2050 INET(I)=0
      DO 52 IZ=1,M1
      IG=NB(IZ)
      DO 1250 IX=1,M1
      IF(INET(IX).EQ.1)GO TO 1250
      IF(INB(J,IX).EQ.IG)GO TO 705
      IFLAG=0
      GO TO 1250
  705 INET(IX)=1
      IFLAG=1
      GO TO 52
 1250 CONTINUE
      IF(IFLAG.EQ.0)GO TO 1200
   52 CONTINUE
      IF(J.GT.N2)NFLAG=1
      RETURN
 1200 J=J+1
      IF(J.LE.MNUM)GO TO 1202
      RETURN
      END
```

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عن مشكلة النقل تحت قيود مختلطة

يتناول هذا البحث مشكلة إيجاد الحل الأمثل لمشكلة نقل ذات دالة هدف خطية ، المراد تقليلها ، خاضعة تحت قيود خطية غير نمطية . فمجموعة القيود في المشكلة المأخوذة في الإعتبار عبارة عن مزيج من المعادلات الخطية واللامتساويات الخطية في الإتجاهين .

وبهذا الشكل الغير نمطي للمشكلة تحولت المشكلة من برنامج لمشكلة نقل يمكن حلها بالطرق الخاصة بحل مشاكل النقل الخطية النمطية ذات الكفاءة الخاصة إلى مشكلة برمجة خطية ذات عدد من القيود الكبير نسبياً . ولقد أمكن تحويل هذا النموذج من مشكلة النقل MP ذات المزيج من القيود المتنوعة إلى مشكلة النقل الخطية النمطية المعروفة ، وهي المشكلة المطلق عليها RTP وفي هذه الورقة أثبتنا في جزء ٢ من خلال نظرية ١ أنه بالحصول على حل أمثل للمشكلة RTP يمكن إستنباط الحل الأمثل للمشكلة الأصلية MP أي أن نظرية ١ تعني ضمنياً أن المشكلة MP يمكن حلها بسهولة وكفاءة بإستخدام أساليب حل مشكلة النقل الخطية النمطية كذلك في جزء ٣ قدمنا طريقة لإيجاد جميع الحلول المثلى ، أن وجدت ، للمشكلة MP وقد بينا من خلال مثال عددي في جزء ٤ مدى أهمية إيجاد جميع الحلول المثلى لمثل هذه المشكلة حيث أن المثال وضح أنه بالرغم من أن جميع الحلول المثلى تعطي نفس القيمة لدالة الهدف إلا أن الحلول قد تختلف في أهميتها بالنسبة لمتخذي القرارات، قد يجوز لغياب إحدى المعايير الغير مأخوذة في الإعتبار في النموذج .

أما في جزء ٥ فقد قدمنا بعض الخبرة الحسابية على الحاسب الآلي PERKIN-ELMER 3220 بمركز المعلومات التخطيطية بالمعهد . وفي الملحق عرضنا برنامج الحاسب الآلي بلغة FORTRAN للطريقة المقترحة .