UNITED ARAB REPUBLIC

THE INSTITUTE OF
NATIONAL PLANNING

التخطيط القومى

Memo. No. 846

ON THE MEASUREMENT OF ECONOMIC GROWTH AN UNBIASED INDEX NUMBER FOR HOMOGENEOUS UTILITY FUNCTIONS

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July, 1968

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I. <u>INTRODUCTION</u>

The last two decades have wittnessed the development of a branch of economic theory which concerns itself with the "Theory of Economic development". Accordingly, the volume of litreture in this area has expanded rapidly. As one might expect, much of the early work in the field dealt with attempts to define and measure economic development, and tried to point out the drawbacks of the alternative definitions and measures. A standard textbook nowadays devotes its first chapter to the discussion of the alternative measures, and concludes that the level of Gross National Product, Net National product, or per capita income can be taken as a measure of material welfare, with some warnings about intertemporal, and inter-country difficulties of comparison. Implicit in some discussions and explicit in others is the notion that rates of growth are more amenable to comparison than the levels of the variables. It is hardly surprising, therefore, to find that policy markers, international organizations, and news media are engaged in inter-temporal, inter-country comparisons of "rates of growth". In this paper, I shall be concerned with one of the problems of measuring the rate of growth, namely the construction of an unbiased index number.

In comparing two positions which differ temporally (or spatially), two quistions arise: (1) Is situation "A" better than "B"? (2) by how much is it better?

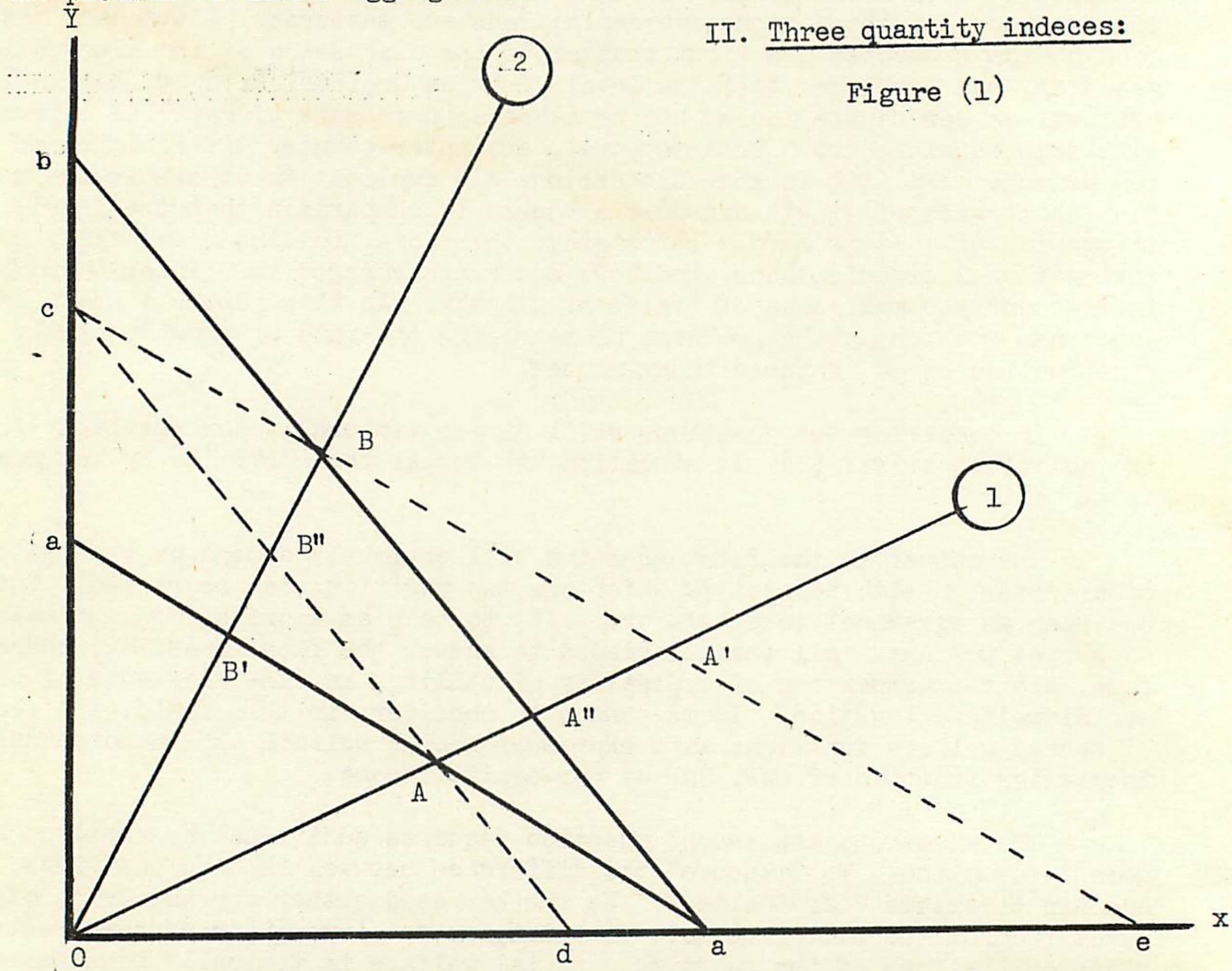
The answer to the first question will obviously depend on the choice of a criterion with respect to which the two positions can be ranked. There has been an agreement (explicit or tacit) to take as a criterion some measure of social welfare. All that is needed to answer the first question, therefore, are the assumption of ordinality of utility, and the knowledge of the social welfare function. It is common to consider, in this field, the class of social welfare functions with expresses social welfare as a monotonically increasing function of GNP, NNP or per capita income.

The answer to the second question requires additional assumptions and more information. To "measure" the difference between the two positions, one has to assume "cardinality". We further need either the knowledge of the exact from of the social welfare function, or the imposition of some restrictions on the form of the function. Social welfare is typically considered as a lineer function in the level of the variables in the context of empirical growth studies, thus enabling us to use the level of the variables as measures of welfare.

We come now to the crux of the problem: constructing a measure of real GNP (or any other variable we select), or an index number for real GNP. This is the familiar problem of constructing a quantity index number. We shall consider three such indeces: Laspeyres, Peasche, and Fisher's Ideal, in Section II, pointing out their difficiencies. In Section III we propose a new index which is characterized by being unbiased, under certain assumptions about the nature of the utility function, and discuss the economic interpretation of the proposed index, and its relation to the problem of linear aggregation.

II. Three quantity indeces:

Figure (1)



Three quantity indeces are most frequently recommended; Laspeyres (L), Paashe (P), and Fisher's Ideal (I). The easiest way to express the relationships between these indeces and to discuss their biases, is through the simplified two commodity case represented in Figure (1) of consumers' equilibria.

Suppose that in period (or country)"0" the budget constraint could be represented by the line aa, and that point A represents the optimal consumption (A is the point of tangency of an indifference curve, which is not shown, and the budget line). The ray (1) represents the composition of consumption in period "0". Similarly, suppose the budget constraint in period "1" can be represented by the line b a, where the point B is the new equilibrium point, and the ray (2) represents the new consumption composition. The dashed line cd is drawn through point A and parallel to ba to represent the budget $\sum Q_o P_1$ and it cuts ray (2) at the point B". The dashed line ce is drawn through point B parallel to aa, to represent the budget $\sum Q_1 P_o$, and it cuts ray (1) at the point A'.

Laspeyres Index number is defined as:

$$L = \sum Q_1 P_o / \sum Q_o P_o = \frac{OA'}{OA} \text{ for the old composition of consumption} \quad (2.1)$$

$$= \frac{OB}{OB'} \text{ for the new composition of consumption}$$

Paasche's Index number is defined as:

$$P = \sum Q_1 P_1 / \sum Q_o P_1 = \frac{OA''}{OA} \text{ for the old composition of consumption} \quad (2.2)$$

$$= \frac{OB}{OB''} \text{ for the newcomposition of consumption}$$

Now to discuss the biases in L and P, let us call the point of intersection of the ray (1) and the social indifference curve (which is tangent to ba at B), A^*; and the point of intersection of ray (2) and the indifference curve (which is tangent to aa at A) B^*. With respect to the point A^* and B^* we have only the following restrictions:

(i) A^* has to be to the Northeast of A" (since there is a tangency at B).

(ii) B^* has to be to the Northeast of B' and the Southwest of B

If the utility function is linear in the quantities consumed, the "Real" Index number can be measured as:

$$R_1 = \frac{OA^*}{OA} \text{ for the old composition of consumption} \quad (2.3)$$

$$R_2 = \frac{OB}{OB^*} \text{ for the new composition of consumption}$$

It is important to notice that, without any further restrictions on the form of the utility function, R_1 need not equal R_2. Our measure of the Real Index number is not inveriant to the choice of consumption composition. To talk about "The Real" index requires additional assumptions which we shall introduce in Part III.

In comparing L and P with R_1 and R_2we shall consider two possibilities:

(1) <u>A^* lies in the interval A"A' and B^* lies in the interval B'B":</u>

$$L = \frac{OA'}{OA} > \frac{OA^*}{OA} = R_1 \qquad (2.4)$$
$$= \frac{OB}{OB'} > \frac{OB}{OB^*} = R_2$$

$$P = \frac{OA''}{OA} < \frac{OA^*}{OA} = R_1 \qquad (2.5)$$
$$= \frac{OB}{OB''} < \frac{OB}{OB^*} = R_2$$

Thus:

$$L > R_1 > P$$
$$L > R_2 > P$$

that is, the two indeces L and P bracket the Real Indeces with respect to both the old and the new composition of consumption.

This simple case has led many writers to suggest an average of L and P as a superior index. The arithmatic mean was suggested by Drobisch, Sidgwick, and recommended by Bowley. The geometric mean was suggested by Walsh, Fisher, Pigou, and Allyn Young. The geometric mean is usually known as Fisher's Ideal Index.

If the conditions obtain such that L and P do bracket the Real indexes, we can write the following relationships:

$$L = R\,(1 + \lambda_1)$$
$$P = R\,(1 + \lambda_P)\,,$$

where R is either R_1 or R_2 and λ_L and λ_P are the biases in L and P. The magnitudes of λ_L and λ_P might change depending on the choice of consumption composition but λ_L will always be positive and λ_p negative.

Fisher's Ideal Index is not unbiased, since:

$$I = \sqrt{L.P} = R\sqrt{(1+\lambda_L)(1+\lambda_P)} \qquad (2.6)$$

(II) <u>A^* Does not lie in the interval A"A' and B^* does not lie in the interval B'B":</u>

In such a case, considering the old composition of consumption, ray (1), we note that:

$$\left.\begin{aligned} L &= \frac{OA'}{OA} < \frac{OA^*}{OA} = R_1 \\ P &= \frac{OA''}{OA} < \frac{OA^*}{OA} = R_1 \end{aligned}\right\} \qquad (2.7)$$

While with respect to the new composition of consumption we get:

$$\left.\begin{aligned} L &= \frac{OB}{OB'} > \frac{OB}{OB^*} = R_2 \\ P &= \frac{OB}{OB''} > \frac{OB}{OB^*} = R_2 \end{aligned}\right\} \qquad (2.8)$$

Both L and P under-estimate the real improvement of the standard of living with respect to the old composition of consumption, and overestimate it with respect to the new composition. Taking an average (arithmatic or geometric) of two biased indeces, where the biases have the same sign, is no improvement.

Combining the results of the two cases, we can write the following inequalities:

a- With respect to the old consumption composition:

$$L \gtrless R_1 > P \qquad (2.9)$$

b- With respect to the new consumption composition:

$$L > R_2 \gtrless P \qquad (2.10)$$

Fisher's Ideal Index will be biased in both cases; the magnitude of its bias relative to the biases in L and P will depend on the nature of the utility function. It will be unbiased only when both L and P are unbiased

($\lambda_1 = \lambda_P = 0$), in which case the Ideal index will be redundant.

The simple arithmatic average of L and P is superior to the Ideal index, in the sense that it is unbiased whenever the Ideal is unbiased, and in addition it will be unbiased if λ_1 and λ_P are of equal magnitude and opposit signs, while the Ideal will still be biased. It will be shown below that the Ideal Index is a special case of the index we propose.

III. An Unbiased Index Number:

It is clear from the preceeding discussion that in the absense of any further information on the nature of the utility function, one can entertain no hope of deriving an unbaised index number. To construct such an index, we seek some constraints on the utility function which will determine the relative size of λ_1 and λ_P. Furthermore, one would like to have the constraints in a form that will allow independent verification.

In this paper we shall impose the constraint that the utility function is homogeneous. While this assumption imply unitary income elasticity of demand on each good, which might be unrealistic, it also implies that the macro-level consumption function is linear homogeneous. This latter implication is supported by the empirical results of studies on the long-run consumption function.

The assumption of homogeneity places some effective constraints on the degree of curveture of the indifference curves between the two rays. In particular, the mapping of such a function will have the property that indifference curves are parallel along any ray from the origin (in economic terms, the marginal rates of substitution are independent of the level of income.)

Now denote the indifference curve which is tangent to aa at A by U_1 and that which is tangent to ba at B by U_2. Homageneity of the utility function guarantees that the slope of U_1 at B^* must be equal to the slope of U_2 at B, which is equal to the slope of the dashed line passing through A. Since the dashed line intersects U_1 at A, and is parallel to U_1 on ray (2), there can be no second intersection in the interval AB"; hence B^* must lie in the interval B'B". Similarly U_1 is parallel to the dashed line passing through B along the ray (1), and they intersect at B, hence A^* must lie in the interval A"A'. The assumption of homogeneous utility function will, therefore, guarantee that L and P bracket the real indeces R_1 and R_2. Furthermore, this assumption guarantees that the Real index is invariant to the choice of consumption composition.

Now consider the homogeneous utility function:

$U = f(x, y)$

For the point A^* to lie on the same indifference curve as the point B, we have the condition:

$$f(x_2, y_2) - f(x^*, y^*) = 0 \qquad (3.1)$$

Furthermore, the point A^* lies on the ray (1) whose equation can be written in the form:

$$y - \frac{y_1}{x_1} x = 0$$

and at point A^* :

$$y^* - \frac{y_1}{x_1} x^* = 0 \qquad (3.2)$$

Substituting for y^* from equation (3.2) in equation (3.1) we get,

$$f(x_2, y_2) - f(x^*, \frac{y_1}{x_1} x^*) = 0 \qquad (3.3)$$

$$f(x_2, y_2) - \frac{x^*}{x_1} f(x_1, y_1) = 0 \qquad (3.3)$$

or

$$R = \frac{f(x^*, y^*)}{f(x_1, y_1)} = \frac{f(x_2, y_2)}{f(x_1, y_1)} = \frac{x^*}{x_1} \qquad (3.4)$$

Suppose that the utility function takes the log-linear form:

$$u = A x^{\alpha} y^{\beta} \qquad \alpha > 0 , \quad \beta > 0 \qquad (3.5)$$

The Real Index number can then be written as:

$$R = f\left(\frac{x_2}{x_1}, \frac{y_2}{y_1}\right) = \left(\frac{x_2}{x_1}\right)^{\alpha} \left(\frac{y_2}{y_1}\right)^{\beta} \qquad (3.6)$$

The equilibrium condition for a maximizing consumer is:

$$\frac{f_x}{f_y} = \frac{P_x}{P_y} = \frac{\alpha}{\beta} \left(\frac{y}{x}\right) = \gamma \left(\frac{y}{x}\right) \qquad (3.7)$$

Substituting in (3.6) using (3.7), we get:

$$R = \left[\frac{\gamma\, y_2\ (P_{y2} / P_{x2})}{\gamma\, y_1\ (P_{y1} / P_{x1})}\right]^{\alpha} \left[\frac{1/\gamma\ (x_2)\ (P_{x2} / P_{y2})}{1/\gamma\ (x_1)\ (P_{x1} / P_{y1})}\right]^{\beta},$$

or

$$R = \left[\frac{y_2\ P_{y2}}{y_1\ P_{y1}}\right]^{\alpha} \left[\frac{x_2\ P_{x2}}{x_1\ P_{x1}}\right]^{\beta} \left[\frac{P_{x1}}{P_{x2}}\right]^{\alpha} \left[\frac{P_{y1}}{P_{y2}}\right]^{\beta} \qquad (3.8)$$

Fisher's Ideal Index turns out to be a special case of (3.8), where we impose the additional constraint:

$\alpha = \beta = \frac{1}{2}$. This can be shown as follows:

$$I^2 = L.P = \frac{(x_2\ P_{x1} + Y_2\ P_{y1})\ (x_2\ P_{x2} + y_2\ P_{y2})}{(x_1\ P_{x1} + y_1\ P_{y1})\ (x_1\ P_{x2} + y_1\ P_{y2})} \qquad (3.9)$$

By multiplication and using the equation $x = \gamma \dfrac{yP_y}{P_x}$ from (3.7), we can write:

$$I^2 = \left[\frac{y_2\ P_{y2}}{y_1\ P_{y1}}\right]\left[\frac{x_2\ P_{x2}}{x_1\ P_{x1}}\right]\left[\frac{P_{x1}}{P_{x2}}\right]\left[\frac{P_{y1}}{P_{y2}}\right]\cdot \left[\frac{\gamma\, P_{x1}\ P_{y2} + P_{y1}\ P_{x2}}{\gamma P_{y1}\ P_{x2} + P_{x1}\ P_{y2}}\right] \qquad (3.10)$$

and

$$I = \left[\frac{y_2\ P_{y2}}{y_1\ P_{y1}}\right]^{\frac{1}{2}} \left[\frac{x_2\ P_{x2}}{x_1\ P_{x1}}\right]^{\frac{1}{2}} \left[\frac{P_{x1}}{P_{x2}}\right]^{\frac{1}{2}} \left[\frac{P_{y1}}{P_{y2}}\right]^{\frac{1}{2}} \left[\frac{\gamma\, P_{x1}\ P_{y2} + P_{y1}\ P_{x2}}{\gamma P_{y1}\ P_{x2} + P_{x1}\ P_{y2}}\right]^{\frac{1}{2}} \qquad (3.11)$$

If $\alpha = \beta = \frac{1}{2}$, then $\gamma = 1$ and I will be equal to R, otherwise, "I" will be biased.

Finally, we seek some technique to estimate R, that is, to estimate α and β , (since the ratios in either (3.6) or (3.8) are observable). Equation (3.7) provides a way to estimate γ . This can be done by regressing $\left(\frac{P_x}{P_y}\right)_t$ on $\left(\frac{y}{x}\right)_t$. If the stochastic element in the observable $\left(\frac{P_x}{P_y}\right)_t$ satisfies the assumptions of zero mean, constant variance, zero temporal convariance and independence of the variable $\left(\frac{x}{y}\right)$ the method of Ordinary Least Squares will yeild the best linear unbiased estimate of γ . If the second and third of these conditions are not satisfied, the Aitken's Generalized Least Squares can be used. If the first assumption is not fulfilled. a constant tearm can be added to the estimating equation. Violation of the fourth assumption would, however, mean that the assumption of homogeneous utility function is inappropriate.

Having obtained the best unbassed linear estimate of γ say $\hat{\gamma}$, we face a problem of identification. We seek to derive two estimates α , β from one equation: $\gamma = \frac{\alpha}{\beta}$. This, clearly cannot be done. We need a second equation in α and β . The degree of homogeneity of the utility function provides such an equation, in the form $\alpha + \beta = \delta$. In the absence of any information on δ , and in the spirit of using GNP as a measure of social welfare, one can impose the restriction: $\alpha + \beta = 1$, and derive the estimates of α and β from that of γ .

It is well known thatthe bias in L and P represent the difference between the "Hicks' income effect" and "Slutskys' income effect". Our discussion makes it possible to distinguish between two components of this bias: (i) Aggregation bias (ii) bias due to the change in relative prices.

Equation (3.8) can be re-written as:

$$R = \left[\left(\frac{y_2\ P_{y1}}{y_1\ P_{y1}}\right)^{\alpha}\left(\frac{x_2\ P_{x1}}{x_1\ P_{x1}}\right)^{\beta}\right]\left[\left(\frac{P_{x1}}{P_{y1}}\right)\Big/\left(\frac{P_{x2}}{P_{y2}}\right)\right]^{\alpha-\beta} \tag{3.12}$$

or

$$R = \left[\left(\frac{y_2\ P_{y2}}{y_1\ P_{y2}}\right)^{\alpha}\left(\frac{x_2\ P_{x2}}{x_1\ P_{x2}}\right)^{\beta}\right]\left[\left(\frac{P_{x1}}{P_{y1}}\right)\Big/\left(\frac{P_{x2}}{P_{y2}}\right)\right]^{\alpha-\beta} \tag{3.12'}$$

The first term in (3.12) and (3.12') contains the elements which appear in L and P respectively. However, the process of aggregation is different. The indeces L and P use linear aggregation, while the proper method, given the utility function assumed, is a weighted log-linear one.

Apart from this source of bias, both L and P ignor the second term of (3.12) and (3.12'). The term involves the change in the relative price. Its omision leads to a bias whose magnitude will depend on the magnitude of change in the relative price of the two commodities.