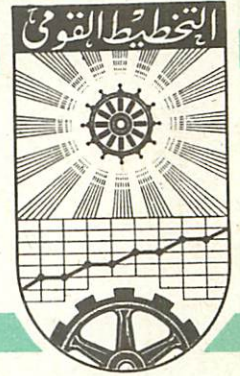


# ARAB REPUBLIC OF EGYPT

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Multi - Parametric Linear  
Programming

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## Introduction:

It is expected that the programming techniques that will be useful to solve different problems in different fields will differ from the classical techniques. Since the increased computer capability will enable to develop techniques which recognize the particular characteristics of the problem being studied. Instead of using some standard techniques which make assumptions about the problem that can not be met. Also A typical manufacturing organization, the various operating departments are for most part goal oriented. Since each department has specific tangible or intangible objectives that are to be optimized conflict may often result. Such problems can be formulated as linear parametric programming. In general various other multiparametric linear programmes may be possible. Linear programmes with multiple parameter either in objective function or in constraints or both are obviously not limited to manufacturing Organizations. Therefore similar motivation could have been given from other areas. It has the advantage of representing many real problems. In anticipation of such approach, it would be quite advantageous to have solutions techniques for such problems. The purpose of our work is to develop a useful technique for solving the multiple parametric linear programming.

1- Multi-parametric linear programming

1-1 Mathematical formulation of linear programming problem:

A linear programming problem can be written as follows:

Find the vector  $X$  that (maximize or minimize) the linear objective function

$$Z' = \underline{C}' X \quad (1)$$

under the constraints

$$\underline{A} X \leq \underline{b} \quad (2)$$

and

$$\underline{X} \geq 0$$

where

$$\underline{A} = A (m, n)$$

$$\underline{b} = b (m, 1)$$

$$\underline{C} = C (m, 1)$$

Equation (1) Form the optimality criterion and define the solution of the problem. i.e the optimal result by the given constraints. System (2) gives the technical, economical and special constraints which bound the solution.

The solution is always seen as a relative solution. The solution of economic problems by such methods is a relative solution, since it can not represent all economic factor in linear programming problem i.e it represents only one side of

that problem. For this reason the optimal solution is relative. A better result, then must be given this result, is obtained from the parametric programming. The parametric optimization programming takes in consideration the main factors of the problem in formulating the model. These main factors are to be taken as parameters and define the objective function with any change in the main factors. From this we develop a useful technique for solving more parametric programming problem.

Another advantage of such parametric programming problem is that programme with more optimal criteria can be represented.

Mathematical formulation of a parametric programming problem is to be done under the following types.

- 1- Parametric objective functions coefficient
- 2- parametric bounds (constraints)
- 3- parametric costs coefficients.

1-2 In the following part I will deal with more-parameteric linear programming there will be four types of problems.

1- For the 1<sup>st</sup> type of those problems will be the optimurn-ing problem as follows

$$Z_1 = \underline{U}' \underline{C}' \underline{X} \quad \text{Maxumum} \quad \underline{U}E.U \quad (1)$$

under the constraints

$$\begin{aligned} \underline{A} \underline{X} &\leq b \\ \underline{X} &\geq 0 \end{aligned}$$

with

$$\underline{U}' = (U_1, U_2, \dots, U_d)$$

and

$$\underline{C}' = \begin{bmatrix} \underline{C}'_1 \\ \underline{C}'_2 \\ : \\ : \\ : \\ \underline{C}'_d \end{bmatrix}$$

The  $\underline{C}'_1, \underline{C}'_2, \dots, \underline{C}'_d$  must be at frinstfree wneights of the paramelers  $U_1, U_2, \dots, U_d$  verfied

the aim is to see

1- For which paraweter branch is an extrem point

an optimal solution

- The properties of the parameter branch
- The properties of Z in this branch
- Which solution can be an optimal solution

the parameters in this sense are free selected

$$(U = E^a) \text{ or limited } (U < E^d).$$

2- the second problem is

$$Z = \underline{C}' \underline{X} \quad \text{maximum} \quad (4)$$

and the constraints

$$\underline{A} \underline{X} \leq \underline{B} \underline{V} \quad \underline{V} \in \underline{V} \quad (5)$$

$$\underline{X} \geq \underline{0}$$

Where the vector  $\underline{b}$  is not given and

$$\underline{b} = \underline{b}_1 V_1 + \underline{b}_2 V_2 + \dots + \underline{b}_g V_g$$

with the parameters

$$V_1, V_2, \dots, V_g$$

then we have

$$\underline{b} = \underline{B} \underline{V}$$

where

$$\underline{B} = (\underline{b}_1, \underline{b}_2, \dots, \underline{b}_g)$$

3- Problem (1) and (2) can be combined together give problem (3) which can be formulated as

$$Z_3 = \underline{U}' \underline{C}' \underline{X} \quad \text{maximum} \quad (7)$$

and

and

$$(U \ E \ U)$$

under the constraints

$$\underline{A} \underline{X} \leq \underline{B} \quad X \quad (V \ E \ V) \quad (8)$$

$$\underline{X} \geq 0 \quad (9)$$

4) The fourth problem

If the Matrix A is not before given, and there is two kinds of Matrices  $\underline{A}^{-1}$  and  $\underline{A}^{-2}$  to be calculated then the problem is as

$$Z_4 = \underline{C}' \underline{X} \quad \max \quad (10)$$

Under the conditions

$$\begin{aligned} (\underline{A}^1 \underline{W}_1 + \underline{A}^2 \underline{W}_2) \underline{X} &\leq \underline{b}, \\ (\underline{W} = \begin{Bmatrix} \underline{W}_1 \\ \underline{W}_2 \end{Bmatrix}) \quad E \quad \underline{W} & \quad (11) \\ \underline{X} &\geq \underline{0} \end{aligned}$$



For problem (1) the more - parameteric Programming problem is

$$\begin{aligned} Z_1 &= \underline{M}' \underline{C}' \underline{X} \quad \text{maximum (UEU)} \\ \underline{A} \underline{X} &\leq b \\ \underline{X} &\geq 0 \end{aligned}$$

the dual of this problem will be

$$\begin{aligned} Z' &= \underline{b}' \underline{y} \quad \text{minimum} \\ \underline{A}' \underline{y} &\geq \underline{C}' \underline{M} \quad \text{(UEU)} \\ \underline{y} &\geq 0 \end{aligned}$$

where

$$\begin{aligned} \underline{C}' &= \underline{C}' (1, n) \\ \underline{U}' \underline{C}' &= (\underline{U}' \underline{C}') (1, n) \end{aligned}$$

as an initial table for the problem

A- the initial table for the problem is

	$\underline{X}'$	
$\underline{y}$	$\underline{A}$	$\underline{b}$
	$\underline{C}'$	$\underline{0} (d, 1)$

the 1<sup>st</sup> table is

	$\underline{X}_1'$	$\underline{X}_2'$	
$Y_1$	$\underline{A}_{11}$	$\underline{A}_{12}$	$\underline{b}_1$
$Y_2$	$\underline{A}_{21}$	$\underline{A}_{22}$	$\underline{b}_2$
$\underline{U}'$	$\underline{c}'_1$	$\underline{c}'_2$	$\underline{0}$

(13)

with which  $\underline{A}_{11} = \underline{A}_{11} \quad (r,r)$  and

$$\underline{c}'_1 = \begin{pmatrix} \underline{c}'_1 \\ \underline{c}'_2 \\ \vdots \\ \underline{c}'_d \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \dots & \vdots \\ c_{d1} & \dots & \dots & c_{dn} \end{pmatrix} = C(d,n)$$

it is also

$$\underline{c}'_1 = \begin{pmatrix} c_{1,J_1} & c_{1,J_2} & \dots & c_{1,J_r} \\ \vdots & \vdots & \dots & \vdots \\ c_{d,J_1} & c_{d,J_2} & \dots & c_{d,J_r} \end{pmatrix} = \underline{c}'_1 (d,r)$$

and

$$\underline{c}'_2 = \begin{pmatrix} c_{1,J_{r+1}} & c_{1,J_{r+2}} & \dots & c_{1,J_n} \\ \vdots & \vdots & \dots & \vdots \\ c_{d,J_{r+1}} & c_{d,J_{r+2}} & \dots & c_{d,J_n} \end{pmatrix} = \underline{c}'_2 (d,n-r)$$

From table (13) through the pivot element  $\underline{A}_{11}^{-1}$  we get the following table

	$\underline{Y}'_1$	$\underline{X}'$	
$\underline{X}_1$	$\underline{A}_{11}^{-1}$	$\underline{A}_{11}^{-1} \underline{A}_{12}$	$\underline{A}_{11}^{-1} \underline{b}_1$
$\underline{Y}_2$	$-\underline{A}_{21} \underline{A}_{11}^{-1}$	$\underline{A}_{22} - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{A}_{12}$	$\underline{b}_2 - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{b}_1$
$\underline{U}'$	$-\underline{C}'_1 \underline{A}_{11}^{-1}$	$\underline{C}'_2 - \underline{C}'_1 \underline{A}_{11}^{-1} \underline{A}_{12}$	$-\underline{C}'_1 \underline{A}_{11}^{-1} \underline{b}_1$

From table (14) is the optimal Solution of the primal problem (1) as

$$\underline{X}^+ = \begin{pmatrix} \underline{A}_{11}^{-1} & \underline{b}_1 \\ & \underline{0} \end{pmatrix} \quad (15)$$

the Solution of the dual problem is

$$\underline{Y}^+ = (\underline{U}' \underline{C}'_1 \underline{A}_{11}^{-1}, \underline{0}') \quad (16)$$

with the following constraints

$$\underline{A}_{11}^{-1} \underline{b}_1 \geq \underline{0} \quad (17)$$

$$\underline{b}_2 - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{b}_1 \geq \underline{0} \quad (18)$$

$$-\underline{U}' \underline{C}'_2 - \underline{A}_{11}^{-1} \leq \underline{0} \quad (19)$$

$$\underline{U}' \underline{C}'_2 - \underline{C}'_1 \underline{A}_{11}^{-1} \underline{A}_{12} \leq \underline{0} \quad (20)$$

the inequalities (19) and (20) are

$$\underline{A}_{11}^{-1} \underline{C}_1 \underline{U}_1 \geq \underline{0} \text{ and}$$

$$(-\underline{C}_2 + \underline{A}_{12} \underline{A}_{11}^{-1} \underline{C}_1) \underline{U} \geq \underline{0}$$

the optimal value of the objective

function of problem (1) is

$$z^+ = \underline{U} \underline{C}_1 \underline{A}_{11}^{-1} \underline{b}_1.$$

Initial Table for solving problem (2):

The problem is given as

$$z_2 = C^1 X \quad \text{maximum ! (21)}$$

under the constraints

$$\underline{A} \underline{X} \leq \underline{B} \underline{V} \quad V \in V, B = B(m, g)$$

$$\underline{V} = \underline{V} (g, 1)$$

$$\underline{X} > 0$$

the dual of this problem is as follows

$$z_2 = \underline{V} \underline{B}' \underline{Y} \quad \text{minimum ! (22)}$$

under the constraints

$$\underline{A}' \underline{Y} \geq \underline{C}$$

$$\underline{Y}' \geq 0$$

as in problem (1) the initial table is as follows:

	$X_1$	$X_2$	$V$
$Y_1$	$\underline{A}_{11}$	$\underline{A}_{12}$	$\underline{B}_1$
$Y_2$	$\underline{A}_{21}$	$\underline{A}_{22}$	$\underline{B}_2$
	$\underline{C}_1'$	$\underline{C}_2'$	$\underline{0} (1, g)$

with the Matrix  $A_{11}$  as a pivot element then table 23 will be in the following form.

	Y1	X2	
X1	$A_{11}^{-1}$	$A_{11}^{-1} A_{12}$	$A_{11}^{-1} B_1$
Y2	$-A_{21} A_{11}^{-1}$	$A_{22} - A_{21} A_{11}^{-1} A_{12}$	$B_2 - A_{21} A_{11}^{-1} B_1$
	$-C_1 A_{11}^{-1}$	$C_2 - C_1 A_{11}^{-1} A_{12}$	$-C_1 A_{11}^{-1} B_1$

the optimal solution of primal problem 21 is

$$X^+ = \begin{pmatrix} A_{11}^{-1} & B_1 & V \\ 0 & & \end{pmatrix}$$

and the optimal solution of the dual

$$Y^+ = (C_1 A_{11}^{-1}, 0)$$

the objective functions of both problem will be

$$Z^+ = C_1 A_{11}^{-1} B_1 + V_1$$

for the optimal solution of 24 it is necessary that

$$A_{11}^{-1} B_1 + V \geq 0 \tag{26}$$

$$(B_2 - A_{21} A_{11}^{-1} B_1) + V \geq 0 \tag{27}$$

$$-C_1 A_{11}^{-1} \leq 0 \quad (28)$$

$$C_2 - C_1 A_{11}^{-1} A_{12} < 0 \quad (29)$$

it must be taken in consideration that since  $V \in B_3$  then 26 and 27 is full filed.

c- Initial table for solving problem (3):

For problem 3 which is

$$Z_3 = \underline{U} \quad \underline{C}' \quad \underline{X} \quad \text{maximum !}$$

$$(\underline{U} \quad E \quad U)$$

under the constraints

$$\underline{A} \quad \underline{X} \quad < \quad \underline{B} \quad \underline{V} \quad (V \quad E \quad V)$$

$$X > 0$$

where

$$\underline{B} = \underline{B} (m, g) \quad \text{with range } g$$

$$\underline{C}' = \underline{C}' (d, n) \quad \text{" " } n$$

the dual of this problem is

$$Z_3 = \underline{V}' \quad \underline{B}' \quad \underline{Y} \quad \text{minimum !}$$

under the constraints

$$\underline{A}' \quad \underline{Y} \geq \underline{C} \quad \underline{U}$$

$$\underline{Y} \geq \underline{0}$$

as an initial table for the solution we put.

	$\underline{x}_1$	$\underline{x}_2$	/ $\underline{v}$
$\underline{y}_1$	$\underline{A}_{11}$	$\underline{A}_{12}$	$\underline{B}_1$
$\underline{y}_2$	$\underline{A}_{21}$	$\underline{A}_{22}$	$\underline{B}_2$
$\underline{u}^1$	$\underline{c}'_1$	$\underline{c}'_2$	$\underline{0}$ (d, g)

(30)

with the change of the variables table 30 will be transformed to the following table

	$\underline{y}_1$	$\underline{x}_2$	$\underline{v}$
$\underline{x}_1$	$\underline{A}_{11}^{-1}$	$\underline{A}_{11}^{-1} \underline{A}_{12}$	$\underline{A}_{11}^{-1} \underline{B}_1$
$\underline{y}_2$	$-\underline{A}_{21} \underline{A}_{11}^{-1}$	$\underline{A}_{22} - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{A}_{12}$	$\underline{B}_2 - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{B}_1$
$\underline{u}$	$-\underline{c}'_1 \underline{A}_{11}^{-1}$	$\underline{c}'_2 - \underline{c}'_1 \underline{A}_{11}^{-1} \underline{A}_{12}$	$-\underline{c}'_1 \underline{A}_{11}^{-1} \underline{B}_1$

the optimal solution of the primal problem of problem 3 will be

$$\underline{x}^+ = \begin{pmatrix} \underline{A}_{11}^{-1} & \underline{B}_1 \underline{v} \\ & \underline{0} \end{pmatrix}$$

and for the dual of this problem.

$$\underline{y}^+ = (\underline{u} \underline{c}'_1 \underline{A}_{11}^{-1}, \underline{0}) \quad (33)$$



where the feasibility of the primal solution required the optimality of the dual. i.e.

$$\begin{aligned} \underline{A}_{11}^{-1} \underline{B}_1 \underline{V} &\geq 0 & \underline{V} \in \underline{V} & \quad (34) \\ (\underline{B}_2 - \underline{A}_{21} \underline{A}_{11}^{-1} \underline{B}_1) \underline{V} &\geq 0 \end{aligned}$$

and also for the optimality of the primal it is required the feasibility of the dual i.e.

$$\begin{aligned} -\underline{U} \underline{C}'_1 \underline{A}_{11}^{-1} &< 0 & \underline{U} \in \underline{U} & \\ \underline{U}' (\underline{C}'_2 - \underline{C}'_1 \underline{A}_{11}^{-1} \underline{A}_{12}) &\leq 0 & & \quad (35) \end{aligned}$$

the objective function of both problems is

$$Z = \underline{U}' \underline{C}'_1 \underline{A}_{11}^{-1} \underline{B}_1 \underline{V} \quad (36)$$

e- Initial table for the problem (4)

for the problem 4 which is in the form

$$Z_4 = \underline{C}' X \quad \text{maximum !}$$

under the constraints

$$(\underline{A}^1 W_1 + \underline{A}^2 W_2) X \leq \underline{b} \underline{W}$$

$$\begin{aligned} W &= \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \in W \\ X &\geq 0 \end{aligned}$$

where

$$W \in \mathbb{R}^2 \quad \text{also } W \neq 0$$

if we examine the columns and rows we get the following initial table

	$\underline{x}_1$	$\underline{x}_2$	
$\underline{y}_1$	$\underline{A}_{11}^1 W_1 + \underline{A}_{11}^2 W_2$	$\underline{A}_{12} W_1 + \underline{A}_{12}^2 W_2$	$\underline{b}_1$
$\underline{y}_2$	$\underline{A}_{21}^1 W_1 + \underline{A}_{21}^2 W_2$	$\underline{A}_{22}^1 W_1 + \underline{A}_{22}^2 W_2$	$\underline{b}_2$
	$\underline{c}'_1$	$\underline{c}'_2$	$\underline{0}$

(37)

with  $\underline{A}_{11}^1 W_1 + \underline{A}_{11}^2 W_2$  as a pivot element (the regularity of the pivot Matrix is necessary) then the above table will be in the form

	$\underline{y}_1$	$\underline{y}_2$	
$\underline{x}_1$	$\underline{A}_{11} (W_1, W_2)$	$\underline{A}_{12} (W_1, W_2)$	$\underline{b}_1 (W_1, W_2)$
$\underline{y}_2$	$\underline{A}_{21} (W_1, W_2)$	$\underline{A}_{22} (W_1, W_2)$	$\underline{b}_2 (W_1, W_2)$
	$\underline{c}'_1 (W_1, W_2)$	$\underline{c}'_2 (W_1, W_2)$	$\underline{z}' (W_1, W_2)$

(38)

where

$$\det (\underline{A}_{11}^1 W_1 + \underline{A}_{11}^2 W_2) \neq 0$$

and with

$$\begin{aligned}
 \bar{A}_{11}(w_1, w_2) &= (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} \\
 \bar{A}_{12}(w_1, w_2) &= (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} (A_{12}^1 w_1 + A_{12}^2 w_2) \\
 \bar{b}_1(w_1, w_2) &= (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} b_1 \\
 A_{21}(w_1, w_2) &= - (A_{21}^1 w_1 + A_{21}^2 w_2) (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} \\
 A_{22}(w_1, w_2) &= (A_{22}^1 w_1 + A_{22}^2 w_2) \\
 &\quad - (A_{21}^1 w_1 + A_{21}^2 w_2) (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} (A_{12}^1 w_1 + A_{12}^2 w_2) \\
 \bar{b}_2(w_1, w_2) &= b_2 - (A_{21}^1 w_1 + A_{21}^2 w_2) (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} b_1 \\
 \bar{C}'_1(w_1, w_2) &= C_1 (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} \\
 \bar{C}'_2(w_1, w_2) &= C_2 - C_1 (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} (A_{12}^1 w_1 + A_{12}^2 w_2) \\
 \bar{z}(w_1, w_2) &= - C_1 (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1} b_1 \tag{40}
 \end{aligned}$$

the optimal solution of the primal problem will be

$$x = \begin{pmatrix} A_{11}^1 w_1 + A_{11}^2 w_2 \\ 0 \end{pmatrix} = \begin{pmatrix} b_1 \\ \end{pmatrix} \tag{41}$$

and for the dual problem the will be

$$\underline{y} = (\bar{C}'_1 (A_{11}^1 w_1 + A_{11}^2 w_2)^{-1}, \underline{0}) \tag{42}$$

with

$$\det (A_{11}^1 W_1 + A_{11}^2 W_2) \geq 0 \quad (43)$$

$$(A_{11}^1 W_1 + A_{11}^2 W_2)^{-1} b_1 \geq 0 \quad (44)$$

also

$$b_2 - (A_{21}^1 W_1 + A_{21}^2 W_2) (A_{11}^1 W_1 + A_{11}^2 W_2)^{-1} b_1 \geq 0$$

$$c_1' (A_{11}^1 W_1 + A_{11}^2 W_2)^{-1} > 0 \quad (45)$$

$$-c_2' + c_1' (A_{11}^1 W_1 + A_{11}^2 W_2)^{-1} (A_{12}^1 W_1 + A_{12}^2 W_2) \geq 0$$

The above mentioned 4 problem can be used for solving many applied problems specially in the field of economic analysis.

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