ARAB REPUBLIC OF EGYPT

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومي

Memo No. (1288).

A Method For Local and Global Minimization of Concave Function Under Linear Constraints

By

Dr. Amani Omar

Feb. 1981.

CAIRO
SALAH SALEM St.-NASR CITY

Introduction

Global optimization involves solving mathematical programming problems that may have distinct local optima. In this paper, I present a method for locating a global minimum (maximum) of a concave (convex) function subjected to linear inequalities. From the mathematical point of view, the concave and convex functions are especially interesting in the theory of nonlinear programming because they have special properties overcoming many difficulties that characterized the nonlinear programming problems. From the economical point of view, a concave function can arise quite easily in economic field because of economies of scale. For in general the cost of production does not increase in proportion to the increase of the number of units produced. In reality, as the number of units produced increases, unit costs decrease. So if $\sum_i C_i X_i$ represents a cost function, where X_i is the number of units of type i produced and C_i is the production cost per unit, then C_i can be approximated by a linear function $d_i + e_i x_i$, where e_i is a negative value, and the cost function $\sum_i C_i X_i = \sum_i (d_i + e_i x_i) X_i$ becomes a concave function.

Similarly, in a competitive situation where an enterprise does not dominate the market place and greater production does not significantly alterprise, then, because of economies of scale, maximizing profits involve maximizing a convex function.

It is well known that any local maximum of a concave function over a closed convex set is also the global maximum. In such a case, that is, a local optimum is also a global optimum, we are not faced with a problem in global optimization. On the other hand, if it is desired to minimize (maximize) instead of maximize (minimize) the concave (convex) function over a convex set, then the problem may have distinct local optima different from the global one, and we are facing the problem of extracting the global optimum out of the local ones. In fact, for any pair m, n, with $m \geq n \geq 3$, one can exhibit n-dimensional problem of a quadratic function and a convex polyhedron for which there are exactly m extreme points all being local optima. The problem is less complicated if the convex set is a polyhedron, for the global and local minimum points of a concave function over a convex polyhedron are taken on at one or more of its extreme points. However, computational procedures so far developed, in general, lead to a solution which is only a local optimum. For example, it would not be possible to use the familiar computational techniques of the simplex type, i.e., based on moving from one extreme point to an adjacent one, since they terminate once a local extreme point-optimum is reached. Moreover, it is usually not possible to determine whether or not the local solution so obtained is really a global optimum. Even if this could be done, no computational algorithm has a way of proceeding from a local optimum to a global optimum. Of course, this is not the case for the linear programming problems, where the simplex method arrives at a solution which is not only a local but also a global optimum.

The first methods for minimizing a concave function over a bounded convex polyhedron have been described in 1964 by Tui (6). His approach to the global optimum was based on the idea of sequentially replacing the problem by subproblems. Zwart gave a three dimensional example in which the sequence of subproblems begins to repeat itself and never ends in the Tui's approach (7). Tui's idea then has been used by some others (see, e.g,4,2).

Ritter (5) gave an algorithm for maximizing any quadratic function subject to linear constraints; his approach based on a sequential reduction of the feasible region, but, Zwart showed that the reduced regions do not approach the empty set and Ritter's algorithm does not converge in general Candler and Townsley (2) have presented an approach to maximizing any quadratic function subject to linear constraints, but, it is heuristic and does not possess the ability to recognize a global maximum. Krynski (4) suggested a method of finding a global minimum of a concave function over a convex polyhedron. The method has a shape of branch-and bound technique and is only a theoretical proposal, The computational efficiency of Krynski's method is poor for problems of even moderate size since a great number of auxiliary problems, which have to be solved, may be generated. In addition, difficulties may arise in the case of degeneracy.

In this paper, an algorithm for maximizing a convex (concave) function over a convex polyhedron is presented. It is computationally finite, does not involve cycling, degenerate situations and unbounded convex polyhedron are considered and treated simply, and all alternate (if there exists more than one) global and local extreme point-optima are generated.

In section I some basic definitions and theorems dealing with concave functions and convex sets are stated; section II contains a brief presentation of Zwart's approach to the problem; the method and its algorithm are described in Sections III and IV; and numerical experience appears in Section V.

I. Basic Definitions and Theorems:

Definition

The function $F(x)$ is said to be concave (convex) over a convex set X in the n-dimensional space R^n if for any two points x_1 and x_2 in X

$$F(\lambda x_2 + (1-\lambda)x_1) \geqslant (\leqslant) \lambda F(X_2) + (1-\lambda) F(X_1),$$

$$\text{for all} \quad 0 \leqslant \lambda \leqslant 1.$$

$F(X)$ is called strictly concave (convex) if the previous relation holds as a strict inequality.

Definition

A function $F(x)$ has a global minimum at a point X_0 of a set $\bar{X}$ if $F(X_0) \leqslant F(x)$ for all x in $\bar{X}$.

Definition

A function $F(X)$ has a local minimum at a point x^o of a set X if there exists a positive number ϵ such that $F(x^o) \leqslant F(X)$ for all x in X at which

$$\|x^o - x\| < \epsilon.$$

of course, a global minimum is also a local minimum, but, the reverse is not true.

Definition

The extreme point X^o of a convex polyhedron X is called a global extreme point-minimum point of $F(x)$ if $F(x^o) \leqslant F(x)$ for any extreme point x of X.

Definition

The extreme point x^o of X is called a local extreme point-minimum point if $F(x^o) \leqslant F(x)$ for any extreme point x of X neighbor to x^o.

Definition

The function F(x) has alternate global minimum points if there exist two or more different points of X where the global unique value of F(x) is taken on.

Definition

The convex hull of a set of points S is the set of all convex combinations of sets of points from S.
For example, the convex polyhedron X is the convex hull of its extreme points.

A set of vectors S is called a cone if for every vector v in S, λv is in S where $\lambda \geqslant 0$.
The cone always contains the origin since λ can equal zero.

Theorem (1)

If F(x) is concave on a convex set X, then F(x) has at most one local maximum which is a global maximum too and is attained on X.

Theorem (2)*

If the global minimum (maximum) of a concave (convex) function F(x) over a convex polyhedron X is finite, then the global minimum (maximum) is taken on at one (or more in case of alternate optima) of the extreme points of X. If F(x) has alternate global optima, then the set of points in X at which F(x) takes on its global value is the convex combinations of the alternate global points.

* For the proof of this theorem see: Hadley ?.: Nonlinear and Dynamic programming. Addison wesley Publishing Company, INC. (1964). P. 83 - 93.

A

11. Zwart's Approach (8):

Zwart has presented an algorithm for the global maximization of a convex function subject to linear inequality constraints. A brief discussion of his approach presented in this section. I have chosen Zwart's method for discussion since it does not involve cycling and it is computationally finite in the following sense: For any prechosen $\epsilon > 0$, a point Z is found in a finite number of steps. It x is any feasible point, then there exists a point y (x) such that $F(y(x)) \leqslant F(Z)$ and $\|x-y(x)\| < \epsilon$.
Finiteness has not been proved for the case $\epsilon = 0$.

The problem is to find x in order to

$$\text{maximize } F(x), \quad \text{subject to } Ax \leqslant d,$$

where A is an mxn real matrix, x and d are column vectors of n and m elements and F(x) is a convex function. It is required to find the global maximum point. It is assumed that the convex polyhedron
$x = \{ x \in R^n : Ax \leqslant d, x \geqslant 0 \}$, is bounded. The method starts by finding an extreme point Z_j of x which is a local maximum for F.
An increasing sequence of compact regions; R_{ji}, i = 1, 2, ... is constructed such that

$$\max \{ F(x) \} = (Z_j), \quad x \in R_{ji} \quad \ldots \quad (1)$$

R_{ji} can be constructed as the convex hull of Z_j and e^i, i = 1, 2, ..., n, where e^i are points found by searching along the adjacent extreme point lines for points that have the same objective value as Z_j. If $x \subseteq R_{ji}$ for some i, then Z_j is a global maximum point. On the other hand, a point $\bar{x}$ of X and not of R_{ji} is located. If $F(\bar{x}) \leqslant F(Z_j)$, then a new set $R_{ji}+1$ that satisfies (1) and contains $\bar{x}$ is constructed. $R_{ji_{+1}}$ can be the convex hull of $\bar{x}$ and R_{ji}. If $F(\bar{x}) > F(Z_j)$, then $\bar{x}$ is used as a starting point to locate Z_{j+1}, a new candidate for the global optimum.

This can be done by searching along the neighbor extreme point lines of $\bar{x}$ till one of the extreme points, Z_{j+1}, which has a better value of F, is found. So Z_{j+1} is a new best local maximum extreme point of X. The whole process is repeated by generating $R_{(j+1)}$'s in step j+1. Computational implementation of Zwart's approach requires a method for constructing the R_{ji}'s and a method for finding points of X that are not in R_{ji} and are local maximum.

For constructing the convex sets R_{ji}, n one-dimensional searches along the neighbor extreme point lines (from Z_j) are carried out to locate the points e^i, $i = 1, \ldots, n$, for which $F(e^i) = F(Z_j)$

Then R_{ji} is constructed as the convex hull of Z_j and e^i.

So for problems of large dimension, i.e., n is large, the search process may take so long time and may be carried out to the farthest distance. Furthermore, for problems having a great number of localoptima or of a global optimum value close to some local points, the number of R_{ji}'s becomes numerous. In addition, if degeneracy is encountered at any extreme point Z_j, it could be very difficult to determine distinct points e^i.

To obtain a new candidate for the global optimum, a number of linear programming problems has to be solved to find points of X that are not in R_{ji}, $j = 1, 2, \ldots$ Each Lp problem has the form:

$$\text{Max } a(e^1, e^2 \ldots, e^n).X \text{ subject to}$$

$X \in X$ and

$$a(e^j, e^2, \ldots, e^{j-1}, e^1, e^{j+1}, \ldots, e^n).(x - e^1) \geq 0,$$

$j = 2, \ldots n+1,$

Where $a(e^1, \ldots, e^n)$ is the unit vector that issuing from e^1 points toward and is perpendicular to the plane determined by $e^2 \ldots e^n$.

This Lp problem is solved to obtain a feasible point $\bar{x}$ of X. This point $\bar{x}$ is a starting point for searching along the neighbor extreme point lines of $\bar{x}$ to find an extreme point that has a better value of F, i.e., a new local maximum Z_{j+1}. Each time a new constraint $a(Z_{j+1}, e^1, \ldots, e^n). X \geq a(Z_{j+1}, e^1 \ldots, e^n). e^1$ is added to the constraints of the current Lp problem to construct a smaller R_{j+1}.

Thus we can see that to locate a new local maximum a number of Lp's must be solved to get feasible starting points. These Lp problems need to be stored by storing the e^j's and keeping track of the proper combination. Since the number of e^i,s are large for large size problems and the number of Lp's grows as the number of local optima increases, then the storage requirements and computation time become critical for large size problems. In addition, the Zwart's method is finite if and only if the sequence of Z_j's is finite and each yields a different value of F. So if there exist alternative global or local maxima the method could have no ability to recognize the global maximum. In other words the method is computationally finite if the global optimum is significantly better than most of the other local maxima, and the local maxima have different values.

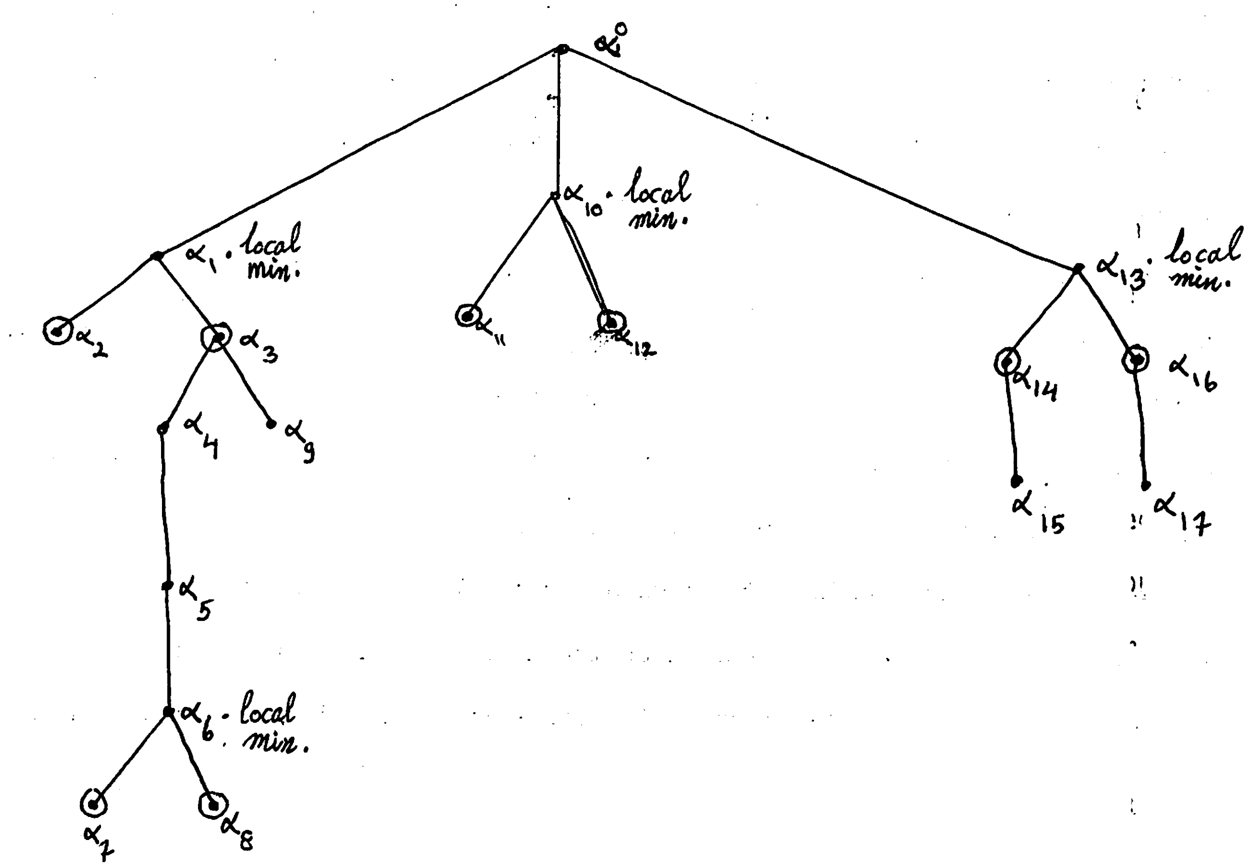
III. A Finite method for the global optimization problem

We are interested in identifying the global minimum of a concave function F(x) over the convex polyhedron $X = \{x \in R^n: Ax \leq d, x \geq 0\}$. The restriction of boundedness of X which is imposed by Zwart's method is not required here in this method, i.e., X could be an unbounded set. The method is based on the standard simplex technique and computationally finite. The following procedure can be applied as well to the problem of maximizing a convex function over X.

We initialize the method by any extreme point of X say, x^o and generate all extreme points neighboring to x^o. The local optimility of x^o is tested by calculating the value of F at x^o and all its neighboring extreme points. If x^o is a local minimum, i.e., $F(x^o) \leq F(x^i)$ for all i=1,2,...,n where x^i are the extreme points neighboring to x^o, then non of x^i could be a local minimum (to see this assume that one of x^i's, say x^j, is a local optimum, then, $F(x^j)$ must be $\leq F(x^o)$ which contradicts the assumption that x^o is a local minimum). Therefore; none of the points x^i will be tested for local optimality. On the other hand if $F(x^o)$ is not a local optimum then any of x^i may be a local minimum and consequently each of x^i i=1, ..., n should be tested for local optimality. We continue the process by finding all new extreme points which are neighbors to each of x^i, i = 1, ..., n. The new extreme points are tested for local optimality if necessary and the whole process is repeated again. It is clear that the process will come to an end in a finite number of steps since the number of the extreme points of X is finite. In case of degeneracy all representations of the same degenerate extreme point must be generated, for distinct new extreme points can be found from the different representations. The number of different representations is finite since it is less than or equal to n times the number of zero basic variables. That is the degenerate cases have no effect on the finitedness of the process.

To locate all extreme points that neighbor any extreme point X°, we examine the nonbasic columns of the simplex tableau corresponding to X° to specify the new points. To ease the programming of the method we may inspect the nonbasic columns of the current simplex tableau in a systematic way either from right to left or from left to right.

The method can be represented by a reee like structure as shown in the following figure for a hypothetical example:



The nodes of the tree represent different extreme points of X. The immediate successors of a node are the distinct new extreme points which neighbor that node. The terminal nodes stand for extreme points which generate no new points. The subscript on a node refers to the simplex iteration at which the extreme point corresponding to that node is produced, if the non-basic columns are scanned from right to left, if the indicators of the new extreme points are kept in order of their appearance, and if the last extreme point found is always chosen for the next iteration. The extreme points corresponding to the nodes in circles are not needed to be examined for local optimality since their immediate predecessors are local minima.

IV. The Algorithm and An Example:

Let the m tuples of unordered integers $\alpha=(i_1, i_2, \ldots, i_m)$, $1 \leq i_j \leq m$, $j = 1, 2 \ldots, m$ be the indices of the basic variables of an extreme point X. We call α the indicator of the point X. Two different extreme points X^1 and X^2 are neighbors if their indicators are different in exactly one component. The algorithm is based on a book keeping of two sets of vectors kept is a matrix W. The dimension of W is (m+1) x r, where r is an upper bound of the number of extreme points of X. The formula used for r is.

$$r = \binom{n+m-\left[\frac{n+1}{2}\right]}{m} + \binom{n+m-\left[\frac{n+2}{2}\right]}{m}^{*}$$

W is divided into two sections; the right section extends from the r-th column to s_i-th column, and the left part extends from the 1-st column

* This formula is given in: McMullen, p.: The maximum Numbers of faces of a convex Polytope. Mathematika, 17(1970).

to S_2-th column. We use the right part to keep the indicators of the extreme points which have been tested for local optimality and the left part to save the indicators of all new neighbors of elements in the right part. The elements of the left part are chosen one by one to be tested for optimality and to create the new neighbors if there remains any.. When the left part of W becomes empty, the right part will contain the indicators of all extreme points of X.*

The steps of the algorithm go as follows:

Step 1: Start with an extreme point X^0 of X and its indicator α^0.
Store α^0 in the r-th column of W. Set s_1=r-1 and S_2=1

Step 2: Test the local optimality of the current extreme point X^0 as follows:

i) For each nonbasic column, determine the pivot row to construct a neighbor indicator of α^0. If the current nonbasic column is nonpositive move to the next one.

ii) Evaluate the values of the concave function for each neighbor extreme point.

iii) Compare the value of F at X^0 with the values calculated at (ii) to find whether X^0 is a local minimum or not.

Step 3: If X^0 is a local minimum, Print $F(X^0)$ and the extreme point X^0.

---

* See: Manas, M. - Nedoma, J.: Finding all vertices of a convex Polyhedron. Numer. Math., 12, (1968).

Step 4: Store the value of F (X^0) in the (m+1)-th location of the α^0- column if X^0 is a local minimum, if not store the letter N.

Step 5: Move from the first column on the left of the current simplex tableau to the next column till the last one on the right to create the indicators neighboring to α^0 :

i) If the inspected nonbasic column is nonpositive go to the next column.

ii) If X^0 is degenerate create all the indicators neighboring to α^0 which associate with the degenerate basic variables.

iii) If X^0 is nondegenerate, determine the pivotal row of the inspected column to construct the neighbor indicator of α^0.

IV) If there are no more columns to be examined, go to step 6.

Step 6: If the indicator(s) created in step 5 is (are) neither in the left nor in the right section of W, then stors it (them) in the S_2-th column (S) of W.

Set $S_2 = S_2 + P$, where P is the number of the new indicators stored. If $S_2 \geqslant S_1$ terminate the program. The available storage of W of W not enough.

Step 7: If X^0 is a local minimum extreme point store the value zero in the (m+1)-th location(S) of the column (S) of the new indicator (S) stored in S_2, otherwise, store the value one.

Step 8: If S_2=0 go to step 11, otherwise, go to step 9.

Step 9: Pick the S_2-th indicator and compute it by carrying out a number of simplex iterations on the X^0- simplex tableau.

Move the S_2-th indicator to the S_1-th column. Set $S_2=S_2-1$ and $S_1 = S_1-1$.

Step 10: If the (m+1)-th location of the (S_1+1)-column has the value zero go to step 5 and if it has the value one go to step 2.

Step 11: Pick the vlaues of the local optima from the (m+1)-th locations of the indicators stored in the right section of W. Compare them with each others to identify the global minimum value.

Print the global minimum value and terminate the process.

A Worked Example:

Let us consider the following illustrative example:

Maximize

$$F(X_1, X_2, X_3) = 25(X_1 - 2)^2 + (X_2-2)^2 + X_3$$

Subject to

$$4X_1 + X_2 + 3X_3 \geqslant 24$$

$$3X_1 + X_2 + 2X_3 \geqslant 4$$

and $X_i \geqslant 0, \quad i=1,2,3$.

The function Z is convex in $(0, +\infty)$.

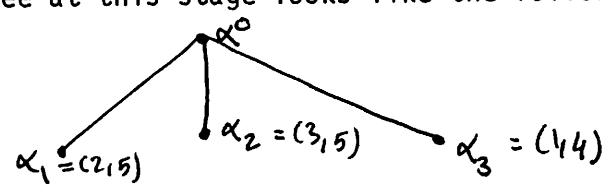
Using the artficial basis technique in the simplex method we get the indicator $\alpha^0 = (1,5)$ and the initial tableau.

	X_2	X_3	X_4	
X_1	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{4}$	6
X_5	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{3}{4}$	14

By examining the nonbasic columns we find that $\alpha_1=(2,5)$, $\alpha_2=(3,5)$ and $\alpha_3=(1/4)$ are neighbor indicators to α^0. Checking the local optimality of $X^0 = (X_1, X_5) = (6,14)$ we find that $F(X^0) = 404 < (X_2, X_5) = 584$. Thus X^0 is not a local optimum. The initial contents of w is

$$\begin{bmatrix} 2 & 3 & 1 & \ldots\ldots & 1 \\ 5 & 5 & 4 & \ldots\ldots & 5 \\ 1 & 1 & 1 & \ldots\ldots & N \end{bmatrix}$$

The indicator) tree at this stage looks like the following.

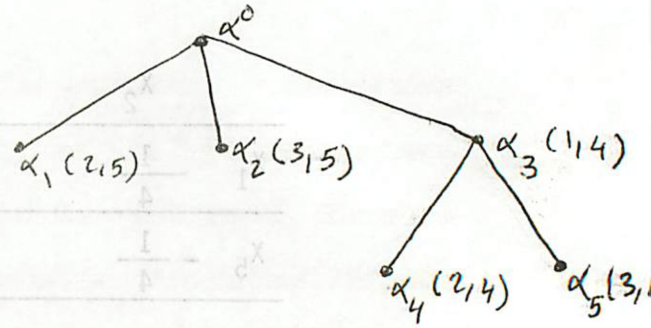


We choose the neighbor indicator α_3= (1 4) and comput it. We get

	X_2	X_3	X_5	
X_1	1/3	2/3	-1/3	4/3
X_4	-1/3	1/3	3/4	56/3

The neighbors are (2,4), (3,4) and (1,5). Checking the local optimality of (X_1, X_4) we find that $F(x_1 x_4) = \frac{136}{9} < F(x_2\ x_4) = 104$, thus, (x_1, x_4) is not a local optimum. The current contents of W and the structure of the indicator-tree are as follows:-

$$\begin{bmatrix} 2 & 3 & 2 & 3 & \ldots\ldots & 1 & 1 \\ 5 & 5 & 4 & 4 & \ldots\ldots & 4 & 5 \\ 1 & 1 & 1 & 1 & \ldots\ldots & N & N \end{bmatrix}$$



We continue by computing the indicator (3,4):

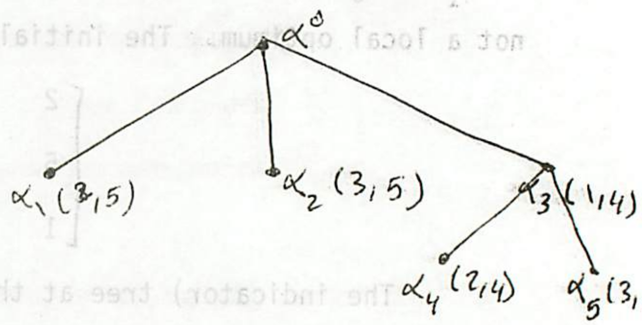
	X_2	X_1	X_5	
X_3	1/2	3/2	1/2	2
X_4	1/2	1/2	3/2	18

$F(X_3, X_4) = 106 < F(X_3, X_5) = 112$

Therefore (X_3, X_4) is not a local optimum.

The current contents of W and the indicator-tree looks as follows:

$$\begin{bmatrix} 2 & 3 & 2 & \ldots\ldots & 3 & 1 & 1 \\ 5 & 5 & 4 & \ldots\ldots & 4 & 4 & 5 \\ 1 & 1 & 1 & \ldots\ldots & N & N & N \end{bmatrix}$$



We continue by computing (2,4):

	X_3	X_1	X_5	
X_2	2	3	-1	4
X_4	1	1	1	20

Since $F(X_2, X_4) < F(X_2, X_5)$, hence, (X_2, X_4) is not a local point.

Since no new extreme points are generated from the last tableau, hence, the contents of W and the tree form are as before. Computing (3,5), We get.

	X_2	X_1	X_4	
X_3	4/3	4/3	1/3	8
X_5	1/3	1/3	2/3	12

The point (X_3, X_5) is not a local optimum since $F(X_2, X_5) > F(X_3, X_5)$.

We compute (X_2, X_5) we find:

	X_3	X_1	X_4	
X_2	3	4	1	24
X_5	1x	1	1	20

Since $F(X_2, X_5) = 584$ is bigger than $F(X_3, X_5)$, and $F(X_1, X_5)$ and $F(X_2, X_4)$, thus,

(X_2, X_5) is a local optimum. Hence the extreme point $(X_1, X_2, X_3, X_4, X_5)$ (0, 24, 0, 0, 20) is also a global optimum for it is a unique local maximum. The final contents of W looks as:

$$\begin{bmatrix} 2 & 3 & 2 & 3 & 1 & 1 \\ 5 & 5 & 4 & 4 & 4 & 5 \\ 584 & N & N & N & N & N \end{bmatrix}$$

In this example we get only one local optimum which is consequently the global optimum. This example serves only to demonstrate the working of the method, but needs not show its power.

V. Numerical Experience:

The previous algorithm has been programmed in FORTRAN VI and used to run a number of examples on the INTERDATA 7/32 computer, computing department, Institute of National Planning. The largest problem solved was of 15 constraints and 10 variables. It took about 31 minutes during which 12 local minima were found. In the appendix we present the results of four test examples; the example of section IV which has been solved by hand and the following :

- Minimize $F(X_1, X_2) = \dfrac{-(X_1-2X_2)^2 + 2X_1 + X_2 + 1}{X_1 + 3X_2 + 1}$

Subject to $X_1 - X_2 \leqslant 2,$

$2X_1 - 5X_2 \leqslant 1$

$-X_1 + 2X_2 \leqslant 0,$

$-2X_1 + 3X_2 \leqslant -1$

and $X_1, X_2 \geqslant 0.$

In this example the function F is concave in R^2, so its global and local optima are located on the extreme points of the polyhedron X defined by the previous constraints. The extreme points of X are $X^1 = (\frac{1}{2}, 0)$, $X^2 = (2,1)$, $X^3 = (3,1)$ and $X^4 = (4,2)$.

The function F takes the value

$F(X^1) = \frac{7}{6}$, $F(X^2) = 1$, $F(X^3) = 1$, $F(X^4) = 1$.

Hence X^2, X^3, X^4 are the optimum solutions.

- Minimize

$$F(X_1, X_2, X_3) = F_1(X_1) + F_2(X_2) + F_3(X_3),$$

$$F_1(X_1) = \begin{cases} 0 & \text{if } X_1 = 0 \\ 2-3X_1 & \text{if } X_1 > 0 \end{cases}$$

$$F_2(X_2) = \begin{cases} -5 & \text{if } X_2 = 0 \\ \dfrac{3X_2+2}{X_2+1} & \text{if } X_2 > 0 \end{cases}$$

$$F_3(X_3) = 3+2X_3 ,$$

Subject to $2X_1 + X_2 - 2X_3 \leqslant 6.$

$X_1 + 2X_2 - 2X_3 \leqslant 7,$

$X_1 - X_2 \leqslant 1.$

and $X_1, X_2, X_3 \geqslant 0.$

The functions F_1, F_2, and F_3 are concave in $[0, +\infty)$, thus their sum is so.

The extreme points are

$X^1 = (0,0,0)$, $X^2 = (1,0,0)$, $X^3 = (7/3, 4/3, 0)$,

$X^4 = (5/3, 8/3, 0)$, and $X^5 = (0, 7/2, 0)$.

The values of F at these points are

$F(X^1) = -2$, $F(X^2) = -3$, $F(X^3) = 4/7$, $F(X^4) = 0\frac{8}{11}$,

and $F(X^5) = 5\ 7/9$.

Hence X^2 is the uniquelocal minimum which is consequently the global solution with $F(X^2) = -3$.

- Maximize $F = 25(X_1 - 2)^2 + (X_2 - 2)^2 + X_3$

Subject to

$$X_1 + X_2 - X_3 = 2,$$

$$4X_2 - X_3 \geqslant 0,$$

$$2X_2 - X_3 \leqslant 4,$$

$$X_1, X_2, X_3 \geqslant 0.$$

The function in this example is convex, so its global maximum is taken at one of the extreme points. This test degenerate problem generates 3 local optima all have the same solutions but with different representations. In fact the degenerate solutions may have the same global value, but the different representations of the same solution may indicate different economic meaning. Thus the degenerate case is worthto be considered. In addition, many of the extreme points can bemissed if not allof the different represenatations be Created.

References

1) Cabot, A.V.: Variations on a cutting plan method for solving concave minimization problems with linear constraints, NRLQ, 21, No.2. (1974).

2) Candter, W. &Townsley, J.: The maximization of a quadratic function subject to linear inequalities, Manag. Sci., No. 10 (1964).

3) Majthay, A. & Whinston, A.: Quasi-concave minimization subject to linear constraints, Discr. Math., No. 9 (1974).

4) Krynski, S.L.: Minimization of a concave function under linear constraints, Polish academy of sciences ul. Newelska 6 (1976).

5) Ritter. "Amethod for solving Maximum Problems with a Nonconcave quadratic objective Function, "Z. Wahrscheinlichkeitstheories uerw. Geb. 4, (1966).

6) Tui, H.: Concave programming under linear Constraints, Dok. Akad, Naul SSR, 159, English (1964).

7) Zwart, P.B.: Nonlinear Programming: Counter examples to global optimization Algorithms by Ritter and Tui, ORSA, 21, No. 4 (1973).

8) Zwart, P.B.: Global maximization of a convex function with linear inequality constraints, ORSA, 22, No.3 (1974).

APPENDIX

```
LOCAL OPT 1   583.99926758
X 2       24.00000000
X 5       20.00000000
VALUE OF GLOBAL OPT. =0.58399927E+03
  1LOCAL OPT.FOUND

END OF PROG.

LOCAL OPT 1      1.00000000
X 1        2.00000000
X 3        1.00000000
X 4        2.00000000
X 2        1.00000000
LOCAL OPT 2      1.00000000
X 1        4.00000000
X 6        1.00000000
X 4        3.00000000
X 2        2.00000000
LOCAL OPT 3      1.00000000
X 1        3.00000000
X 6        2.00000000
X 5        0.99999994
X 2        1.00000000
VALUE OF GLOBAL OPT =0.10000000E+01
  3LOCAL OPT FOUND

END OF PROG.

LOCAL OPT 1     -3.00000000
X 1        1.00000000
X 4        4.00000191
X 5        6.00000191
VALUE OF GLOBAL OPT.-0.29999997E+10
  1LOCAL OPT.FOUND

END OF PROG.

LOCAL OPT 1      4.00000000
X 1        2.00000000
X 5        4.00000000
X 6        4.00000000
X 7        0.00000000
LOCAL OPT 2      4.00000000
X 1        2.00000000
X 5        4.00000000
X 6        4.00000000
X 3        0.00000000
LOCAL OPT 3    100.00000000
X 2        2.00000000
X 3        0.00000286
X 6        4.00000286
X 7        8.00000000
LOCAL OPT 4      8.00001526
X 2        4.00000381
X 1        2.00000095
X 3        4.00000000
X 7       12.00001049
LOCAL OPT 5      3.99999619
X 2        0.00000095
X 1        2.00000095
X 6        4.00000000
X 5        4.00000381
VALUE OF GLOBAL OPT. =0.10000000E+03
  5LOCAL OPT.FOUND

END OF PROG.
```