UNITED ARAB REPUBLIC

THE INSTITUTE OF NATIONAL PLANNING

التخطيط القومي

Memo. No. 423

A NOTE ON LIFE OF CAPITAL AND ECONOMIC GROWTH

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April 9, 1964.

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A NOTE ON LIFE OF CAPITAL AND ECONOMIC GROWTH.[§]

At least three procedures of factor combination can be distinguished in the aggregative growth models that allow for continuous factor substitution along a course set by the particular production function used. One of them was perhaps initiated by Prof. Tinbergen (1) followed by Haavelmo (2) Solow (3) and Valavanis-Vail (4). This procedure assumes 'explicitly expressed possibilities of substitution between the total amounts of factors available,' that is, as the total amount of factors grow, they are simultaneously combined in their entirety with each other. For example, if we confine ourselves to two factors of production, labour and capital, and denote their total amounts in a period t by $\bar{L}_t$ and $\bar{K}_t$, then the total output in period t

$$\bar{P}_t = F(\bar{L}_t, \bar{K}_t),$$

where F stands for the production function, which we will assume to be linear homogeneous.

The marginal productivity of factors are

$$\bar{w}_t = \frac{\partial F(\bar{L}_t, \bar{K}_t)}{\partial \bar{L}_t}$$

$$\bar{r}_t = \frac{\partial F(\bar{L}_t, \bar{K}_t)}{\partial \bar{K}_t}$$

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§ I am grateful to Prof. Bent Hansen and Mr. C.J. van Rees who have gone through the manuscript. I am indebted to Messrs. Abdulla Bin Amir and M.A.M. Nassar for working out the proof contained in this note.

We shall call $\bar{w}$ and $\bar{r}$ over-all marginal productivity of labour and capital respectively in view of the fact that they are derived through the procedure involving substitution of total labour and total capital. However, this procedure is feasible only then (1) the life of capital built is one period only or (2) the factors of production have been growing at the same rate for a long time so that the ratio of the marginal increases in their amounts is equal to the ratio of their total amounts. These conditions are seldom fulfilled, hence this procedure is, in most cases, not feasible.

The second procedure of factor combination is that which has been adopted by Johansen (5) followed by Massell (6). The basic assumption underlying this approach is that 'there are substitution possibilities ex-ante, but not ex-post', i.e. 'any gross increment in the rate of production can be obtained by different combinations of increments in capital and labour input'.[1] The procedure followed by these writers combines gross investment in each period with the uncommitted labour supply in that period, thus ensuring full employment of both the factors. We shall call gross investment and uncommitted supplies of labour fresh supplies of labour and capital and denote them by $\tilde{L}$ and $\tilde{K}$ respectively. The (gross) increment of output in a period would accordingly be

$$\tilde{P}_t = F(\tilde{L}_t, \tilde{K}_t)$$

and the resulting marginal productivity of factors

$$\tilde{w}_t = \frac{\partial F(\tilde{L}_t, \tilde{K}_t)}{\partial \tilde{L}_t}$$

$$\tilde{r}_t = \frac{\partial F(\tilde{L}_t, \tilde{K}_t)}{\partial \tilde{K}_t}$$

We call $\tilde{w}$ and $\tilde{r}$ <u>incremental</u> marginal productivity of labour and capital respectively. If the amounts of labour and capital, $\tilde{L}_t$ and $\tilde{K}_t$, are combined according to this procedure the resulting output in each period during the life-time of the capital built would be

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1) Leif Johansen (5, p.158).

$$\tilde{P}_t = \tilde{L}_t \tilde{w}_t + \tilde{K}_t \tilde{r}_t .$$

If the capital built lasts θ periods, the total output resulting from $\tilde{L}_t$ and $\tilde{K}_t$ would be $\theta \tilde{P}_t$.

It is obvious that the second procedure is feasible, but it is contended here that it may be inefficient. It is so because there exists a third feasible procedure of factor combination which may result in a higher level of output from the same amount of factors.

The basic idea behind the procedure to be suggested here is that the combination of fresh supplies of factors should not reflect the relative scarcity of factors in the current period only, but also the relative scarcities of their accumulated amounts in the succeeding periods till the last period of the life of capital built. This procedure is feasible as it is possible to decide any combination of factors ex-ante, i.e., before the capital has been built as will be readily agreed to. Thus if we can anticipate the amounts of fresh supplies of factors, we can choose a technique which is not only compatible with the fresh supplies of factors in the current period but also with their accumulated amounts in the succeeding periods. We now define what we shall call cumulative marginal productivity of factors. Let us suppose that we are concerned with choosing a technique in period 1, and that the life of capital to be built is θ periods, the cumulative marginal productivity of labour and capital in period $n (1 \leq n \leq \theta)$ is

$$w^*_n = \frac{\partial F(L^*_n, K^*_n)}{\partial L^*_n} \qquad r^*_n = \frac{\partial F(L^*_n, K^*_n)}{\partial K^*_n}$$

where $L^*_n = \tilde{L}_1 + \tilde{L}_2 + \ldots + \tilde{L}_n$ $\qquad K^*_n = \tilde{K}_1 + \tilde{K}_2 + \ldots \ldots + \tilde{K}_n$.

Now if we visualize a situation in which the same amounts of $\tilde{L}_1$ and $\tilde{K}_1$ of fresh supplies of labour and capital are combined according to the cumulative marginal productivity in period 1, and the life of capital built is θ periods, then the total output over Θ periods will be[1)]

$$\Sigma P_1^* = \tilde{L}_1(w_1^* + w_2^* + \ldots + w_\Theta^*) + \tilde{K}_1(r_1^* + r_2^* + \ldots + r_\Theta^*)$$

$$\Sigma P_1^* = \tilde{L}_1 \sum_1^\Theta w_n + \tilde{K}_1 \sum_1^\Theta r_n$$

What we have to show now is that $\Sigma P_1^* \geq \Theta \tilde{P}_1$ where $\tilde{L}_1$ and $\tilde{K}_1$ are the same in the two procedures. [This will imply that if $\Sigma P_1 = \Theta \tilde{P}_1$, then at least one of the factors used in case of the factor-combination according to the cumulative marginal productivity of factors will be smaller than that will be required according to the incremental marginal productivity of factors.]

In order to prove that $\Sigma P_1^* \geq \Theta \tilde{P}_1$, we assume that fresh supplies of labour and capital grow at constant rates λ and k ($\lambda \geq 0$, $k \geq 0$) respectively, so that

$$L_t = L_1 e^{(t-1)\lambda} \quad \text{and } K_t = K_1 e^{(t-1)k}.$$

1) When Θ is equal to the number of factors, the relative amounts of factors compatible with this procedure is exactly determined. In general Θ will be greater than the number of factors, in which case the system will be over-determined. We, therefore, do not require

$$P_1^* = \tilde{L}_1 w_1^* + \tilde{K}_1 r_1^* = \tilde{L}_1 w_2^* + \tilde{K}_1 r_2^* = \ldots\ldots = \tilde{L}_1 w_n^* + \tilde{K}_1 r_n^* =$$

$$= \ldots = \tilde{L}_1 w_\Theta^* + \tilde{K}_1 r_\Theta^*$$, but an equality in the aggregate as given in the text.

It is not always possible to employ all the fresh supplies of factors instantaneously according to the cumulative marginal productivity. The reader may visualize a situation in another country where the availability of fresh supplies of factors over the coming Θ periods are such that L_1 and K_1 can be combined in period 1 according to the cumulative marginal productivity.

We further assume that F is a Cobb-Douglas type, i.e.,

$P = L^{\alpha} K^{\beta}$, $\alpha + \beta = 1$. Then

$$\Sigma P_1^{*} = \alpha \tilde{L}_1^{\alpha-1} . \tilde{K}_1^{\beta} \tilde{L}_1 \Big\{ 1+(1+e^{\lambda})^{\alpha-1} (1+e^{k})^{\beta} + (1+e^{\lambda}+e^{2\lambda})^{\alpha-1} .$$

$$(1+e^{k}+e^{2k})^{\beta} + \ldots + (1+e^{\lambda}+e^{2\lambda}+ \ldots + e^{(\theta-1)\lambda})^{\alpha-1} (1+e^{k}+e^{2k}+ \ldots$$

$$\ldots + e^{(\theta-1)k})^{\beta} \Big\} + \beta \tilde{L}_1^{\alpha} \tilde{K}_1^{\beta-1} . \tilde{K}_1 \Big\{ 1+(1+e^{\lambda})^{\alpha} .$$

$$(1+e^{k})^{\beta-1} + (1+e^{\lambda}+e^{2\lambda})^{\alpha} . (1+e^{k}+e^{2k})^{\beta-1} + \ldots + (1+e^{\lambda}+e^{2\lambda} \ldots e^{(\theta-1)\lambda})^{\alpha} .$$

$$(1+e^{k}+e^{2k} + \ldots + e^{(\theta-1)\lambda})^{\beta-1} \Big\}$$

and $$\theta \tilde{P}_1 = \theta \left[\tilde{L}_1 \alpha \tilde{L}_1^{\alpha-1} \tilde{K}_1^{\beta} + \tilde{K}_1 \beta \tilde{L}_1^{\alpha} \tilde{K}_1^{\beta-1} \right].$$

It is obvious that the terms corresponding to the first term in both ΣP_1^{*} and $\theta \tilde{P}_1$ are equal. The difference between the terms corresponding to the second period in ΣP_1^{*} and $\theta \tilde{P}_1$ is

$$\alpha \tilde{L}_1^{\alpha} \tilde{K}_1^{\beta} \Big\{ (1+e^{\lambda})^{\alpha-1} . (1+e^{k})^{\beta} - 1 \Big\} + \beta \tilde{L}_1^{\alpha} \tilde{K}_1^{\beta} \Big\{ (1+e^{\lambda}) . (1+e^{k})^{\beta-1} - 1 \Big\} \quad \ldots (1)$$

where $\alpha + \beta = 1$.

Let us suppose that (1) is negative. This will be the case when

$$\alpha \left[\left\{ \frac{(1+e^{\lambda})}{(1+e^{k})} \right\}^{\alpha-1} - 1 \right] + (1-\alpha) \left[\left\{ \frac{(1+e^{\lambda})}{(1+e^{k})} \right\}^{\alpha} - 1 \right] < 0 \quad \ldots\ldots (2)$$

Let $\frac{1+e^{\lambda}}{1+e^{k}} = 1 + \epsilon$ where ϵ is a small number $\gtreqless 0$ according as $\lambda \gtreqless k$.[1)]

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1) As each of λ and k are positive and in reality cannot exceed .04 and .05, this assumption is justified.

Ignoring terms containing higher powers of ϵ than ϵ^2, (2) will hold only if $\alpha - 1 < \alpha - 2$. As this is impossible, (2) cannot be negative. It must be non-negative, zero when $\lambda = k$ and positive when $\lambda \neq k$.

In order to prove that the same is true for the difference between the respective terms in ΣP_1^* and $\Theta \tilde{P}_1$ corresponding to the following periods, we use the induction method. We have to show that if the difference between the term corresponding to the n_{th} period in ΣP_1^* and that in $\Theta \tilde{P}_1$ is non-negative then the same is true for the $(n+1)_{th}$ period, i.e., if

$$\alpha (Q^{\alpha-1} - 1) + \beta (Q^{\alpha} - 1) > 0 \quad \ldots\ldots (3)$$

then

$$\alpha \left\{(TQ)^{\alpha-1} - 1\right\} + \beta \left\{(TQ)^{\alpha} - 1\right\} > 0 \quad \ldots\ldots (4)$$

where

$$Q = \frac{1+e^{\lambda} + e^{2\lambda} + \ldots + e^{(n-1)\lambda}}{1+e^{k} + e^{2k} + \ldots + e^{(n-1)k}}$$

and

$$TQ = \frac{1+e^{\lambda} + e^{2\lambda} + \ldots + e^{(n-1)\lambda} + e^{n\lambda}}{1+e^{k} + e^{2k} + \ldots + e^{(n-1)k} + e^{nk}}$$

(3) can be reduced to

$$\alpha + \beta Q > Q^{\beta} \quad \ldots\ldots (5)$$

and (4) to

$$\alpha + \beta TQ > (TQ)^{\beta} \quad \ldots\ldots (6)$$

For a non-trivial proof, put $Q = 1 + \epsilon'$ and $T = 1 + \delta$, where the signs of ϵ' and δ are similar. Then (6) will be proved, if we can prove the following

$$\alpha + \beta(1+\delta)(1+\epsilon') \geq (1+\delta)^{\beta}\{\alpha+\beta(1+\epsilon')\} \geq (1+\delta\beta)(\alpha+\beta+\beta\epsilon') \quad \ldots\ldots \quad (7)$$

(remembering $\beta < 1$, so that the third term in the expansion of $(1+\delta)^{\beta}$ is negative and, δ being small, the successive terms go on decreasing with alternative signs).

(7) is reduced to

$$\beta\epsilon'\delta \geq \beta^2\epsilon'\delta$$

and as the signs of ϵ' and δ are similar, we have

$$1 \geq \beta.$$

As the above is obvious, hence the proof.

This means that if the rates of growth of labour and capital are the same, the factor combination according to Johansen-Massell approach and the one suggested here will be the same and the level of resultant output will also be the same. And if the rates of growth of labour and capital have been equal for a sufficiently long time in the past, then Tinbergen's procedure of factor combination will also be the same as the other two. These results are clear intuitively as well. If the factors are combined according to the incremental marginal productivity of the current period, they will reflect the relative scarcity of the fresh supplies of factors in that period only, but will fail to reflect the changes in the relative scarcity of factors as they get accumulated. Hence the need for an alternative approach which takes into account the changing scarcity of factors.

For instance if a plant is going to last about 25 years, it will be evidently inadvisable and inefficient to adopt a technique which only reflects the current scarcity of the fresh supplies of factors. On the contrary, the techniques should be such that they reflect the expected changes in the relative scarcity of factors over the coming 25 years.

It should be noted that the cumulative approach does not ensure instantaneous full employment as is the case with Johansen approach, but it precludes obsolescence as discussed by Kurz (7).[1] In a capital scarce economy, obsolescence exceeding a certain minimum unavoidable limit is clearly wasteful.

1) I have come across Kurz's article after preparing the draft of this note.

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